

CO-DOUBLE BOSONISATION AND DUAL BASES OF $c_q[SL_2]$ AND $c_q[SL_3]$

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ABSTRACT. We find a dual version of a previous double-bosonisation theorem whereby each finite-dimensional braided-Hopf algebra B in the category of comodules of a coquasitriangular Hopf algebra A has an associated coquasitriangular Hopf algebra $B^{\text{op}} \bowtie A \bowtie B^*$. As an application we find new generators for $c_q[SL_2]$ reduced at q a primitive odd root of unity with the remarkable property that their monomials are essentially a dual basis to the standard PBW basis of the reduced Drinfeld-Jimbo quantum enveloping algebra $u_q(sl_2)$. Our methods apply in principle for general $c_q[G]$ as we demonstrate for $c_q[SL_3]$ at certain odd roots of unity.

1. INTRODUCTION

Quantum groups such as $U_q(\mathfrak{g})$ associated to complex semisimple Lie algebras [3, 5], and their finite-dimensional quotients $u_q(\mathfrak{g})$ at q a primitive n -th root of unity, have been extensively studied since the 1980s and 1990s respectively. The latter are covered in several texts such as [7, 4], although precise definitions and the relation to Lusztig's celebrated divided-difference versions of $U_q(\mathfrak{g})$ are quite subtle and depend on the precise root when n is small. See [6] for a recent work. There are also corresponding 'coordinate algebra' quantum groups $C_q[G]$ and in principle reduced finite-dimensional quotients $c_q[G]$ although again best understood for specific cases [2].

In spite of some extensive literature, one problem which we believe to be open till now even for the simplest case of $c_q[SL_2]$ is a description of its dual basis in terms of the generators and relations. Here $u_q(sl_2)$ has generators F, K, E with the relations of $U_q(sl_2)$ and additionally $E^n = F^n = 0, K^n = 1$, and PBW basis $\{F^i K^j E^k\}$ for $0 \leq i, j, k < n$. The dual Hopf algebra $c_q[SL_2]$ is a quotient of $\mathbb{C}_q[SL_2]$ with its standard matrix entry generators a, b, c, d , and the additional relations $a^n = d^n = 1, b^n = c^n = 0$ to give a Hopf algebra extension

$$\mathbb{C}[SL_2] \hookrightarrow \mathbb{C}_q[SL_2] \twoheadrightarrow c_q[SL_2].$$

This $c_q[SL_2]$ has an obvious monomial basis $\{b^i a^j c^k\}$ but its Hopf algebra pairing with the PBW basis of $u_q(sl_2)$ is rather complicated (it can be related to the representation theory of the quantum group) and this does not constitute a dual basis even up to normalisation. Knowing a basis and dual basis is equivalent to knowing the canonical coevaluation element, which has many applications including Hopf algebra Fourier transform. Here we solve the dual basis problem for $c_q[SL_2]$ at q a primitive odd root of unity in Corollary 4.3, finding new generators X, t, Y of $c_q[SL_2]$ such that normalised monomials $\{X^i t^j Y^k\}$ are essentially a dual basis in the sense of being dually paired by

$$\langle X^i t^j Y^k, F^{i'} K^{j'} E^{k'} \rangle = \delta_{ii'} \delta_{kk'} q^{jj'} [i]_{q^{-1}}! [k]_q!,$$

where $[i]_q$ etc. are q -integers. An actual dual basis immediately follows. Section 6 similarly computes the dual basis for $c_q[SL_3]$ at certain roots of unity including all n that are prime and congruent to $\pm 1 \pmod{12}$.

This result depends on a general 'braided quantum double' or *double-bosonisation* construction [10, 11, 15] which associates to each finite-dimensional braided-Hopf algebra ('braided group') B living in the category of modules over a quasitriangular Hopf

algebra H , a new quasitriangular Hopf algebra

$$B^{*\text{cop}} \bowtie H \bowtie B =: D_H(B),$$

where the second notation has also been used in the literature in line with the view of this in [11] as the closest one can come to the bosonisation of a ‘braided double’ of B (the latter does not itself exist in the strictly braided case). Thus, for any n we take $H = \mathbb{C}[K]/(K^n - 1)$ with its natural quasitriangular structure \mathcal{R}_K so that its modules are the category of \mathbb{Z} -graded spaces with braiding given by a power of q according to the degrees. We take $B = \mathbb{C}[E]/(E^n)$ and $B^{*\text{cop}} = \mathbb{C}[F]/(F^n)$ and obtain a version

$$B^{*\text{cop}} \bowtie H \bowtie B = \mathfrak{u}_q(sl_2) \cong \begin{cases} u_p(sl_2) & n = 2m + 1, p = q^{-m}; p^2 = q, \\ \text{something else} & n \text{ even.} \end{cases}$$

To illustrate the even case, $\mathfrak{u}_{-1}(sl_2)$ in Example 4.4 is an interesting 8-dimensional strictly quasitriangular and self-dual Hopf algebra presumably known elsewhere. Our approach to the dual basis problem is to work out the dual version or *co-double bosonisation* and use this to construct the dual of $\mathfrak{u}_q(sl_2)$ in the dual form

$$B^{\text{op}} \bowtie A \bowtie B^* = \text{co}D_A(B),$$

where each tensor factor pairs with the corresponding factor on the $\mathfrak{u}_q(sl_2)$ side. This dual version of double bosonisation is in Section 3 and is conceptually given by reversing arrows in the original construction, but in practice takes a great deal of care to trace through all the layers of the construction. Moreover, we do not want to be limited to finite-dimensional A and give a self-contained algebraic proof for A any co-quasitriangular Hopf algebra and B a finite-dimensional braided group living in the category of its comodules. When A is also finite dimensional, the resulting object has a Hopf algebra duality pairing with the double bosonisation by, schematically, $\langle B^{\text{op}} \bowtie A \bowtie B^*, B^{*\text{cop}} \bowtie H \bowtie B \rangle = \langle B^{\text{op}}, B^{*\text{cop}} \rangle \langle A, H \rangle \langle B^*, B \rangle$. Unlike double bosonisation, the co-double bosonisation has coalgebra surjections to the constituents B^{op}, A, B^* making calculations for it harder than the original version.

In Section 4 we take $B = \mathbb{C}[X]/(X^n)$ and its dual $B^* = \mathbb{C}[Y]/(Y^n)$, again braided-lines but this time viewed in the category of $A = \mathbb{C}[t]/(t^n - 1)$ -comodules with its standard coquasitriangular structure $\mathcal{R}(t, t) = q$ (so that its comodules form the same braided category of \mathbb{Z} -graded vector spaces as before). Then the co-double bosonisation gives a coquasitriangular Hopf algebra $\mathfrak{c}_q[SL_2] = c_p[SL_2]$ when n is odd and some other coquasitriangular Hopf algebra when n is even. This is Theorem 4.1 with the dual basis result a corollary of the triangular decomposition. As an application, Hopf algebra Fourier transform $\mathcal{F} : \mathfrak{c}_q[SL_2] \rightarrow \mathfrak{u}_q(sl_2)$ is worked out in Section 5 and shown to behave well with respect to the 3D-calculus of $c_p[SL_2]$. Another application of the canonical element for the pairing, which we do not discuss, is to provide the quasitriangular structure of the Drinfeld double $D(\mathfrak{u}_p(sl_2))$ of interest in 3D quantum gravity.

Although the details rapidly become complicated, Sections 6.1 and 6.2 similarly study the next iteration, $\mathfrak{c}_q[SL_3]$ as dually paired to $\mathfrak{u}_q(sl_3)$, where $H = \widetilde{\mathfrak{u}_q(sl_2)}$ is a certain central extension and $B = \mathfrak{c}_q^2$ denotes the usual quantum-braided plane but reduced at the root of unity by making the generators nilpotent of order n . The central extension requires an integer β such that $\beta^2 = 3 \pmod n$, where we assume that $n = 2m + 1$ is odd and set $p = q^{-m}$. The dual of B has a similar form and double-bosonisation gives us

$$\mathfrak{u}_q(sl_3) \cong \begin{cases} u_p(sl_3) & m > 1, \\ (u_p(sl_3)/(K_1 - K_2)) \otimes (\mathbb{C}[g]/(g^n - 1)) & m = 1, \end{cases}$$

where the $m = 1$ case equates the two Cartan generators of the usual quantum group. This quotient is necessarily quasitriangular by our construction whereas we are not clear if this is the case for $u_p(sl_3)$ itself when $q^3 = 1$. We then construct the dual by $A = \widetilde{\mathfrak{c}_q[SL_2]}$ and similar quantum-braided planes now as Hopf algebras in its category comodules lead to a dual coquasitriangular Hopf algebra $\mathfrak{c}_q[SL_3]$. For $m > 1$ we show

that this is isomorphic to the usual $c_p[SL_3]$ while for $m = 1$ we obtain a central extension of a sub-Hopf algebra of $c_p[SL_3]$. Clearly, one could go on to analyse other choices of n , as well as to look similarly at the next iteration for $u_q(sl_4)$ and its duality with $c_q[SL_4]$, etc. Even at the second stage of $H = \widetilde{u_q(sl_2)}$, there are other potential choices for braided planes including some that give versions of $u_q(g_2)$ and $u_q(sp_2)$ (details will be given elsewhere) and others that have no classical picture at all. Existence was covered at the semiclassical or Lie bialgebra level in [13] as an inductive process that adds one to the rank of the Lie algebra at each iteration, and is also clear for generic $U_q(\mathfrak{g})$ in a suitable setting [10]. Section 6.3 illustrates a non-classical choice where $A = \mathbb{C}_q[GL_2]$ is not finite dimensional, q is generic and $B = \mathbb{C}_q^{0|2}$ is the ‘fermionic quantum-braided plane’ in the category of A -comodules. This leads to an exotic but still coquasitriangular version of $\mathbb{C}_q[SL_3]$ with some matrix entries ‘fermionic’. We also note that the inductive approach, even after multiple iterations, preserves a triangular decomposition in which the accumulated central generators form the ‘Cartan’ factor, the accumulated braided groups B form a ‘positive’ braided group on one side and their duals form a ‘negative’ braided group on the other side. For the classical families, this recovers versions of $u_q(n_\pm)$ but now as braided-Hopf algebras with dual bases.

2. PRELIMINARIES

We recall the notations and facts about Drinfeld’s (co)quasitriangular Hopf algebras as can be found in several texts, for example [14, 15], braided groups and bosonisation as introduced in [17, 18, 19, 14] and double bosonisation [10, 11, 12, 15]. We also establish lemmas needed for a clean presentation of the latter and its dualisation.

2.1. Quasitriangular Hopf algebras. Recall that a Hopf algebra is (H, Δ, ϵ, S) where H is a unital algebra, $\Delta : H \rightarrow H \otimes H$ and $\epsilon : H \rightarrow k$ form a coalgebra and are algebra maps, and there is an antipode $S : H \rightarrow H$ obeying $(Sh_{(1)})h_{(2)} = \epsilon(h) = h_{(1)}(Sh_{(2)})$ for all $h \in H$. We use Sweedler’s notation $\Delta h = h_{(1)} \otimes h_{(2)}$ (summation understood) and k is the ground field. Modules/comodules of H have a tensor product defined respectively by pull back/push out along the co/product. Another Hopf algebra A is ‘dually paired’ if $\langle \cdot, \cdot \rangle : A \otimes H \rightarrow k$ makes the coalgebra and antipode on one side adjoint to the algebra and antipode on the other (e.g., $\langle ab, h \rangle = \langle a, h_{(1)} \rangle \langle b, h_{(2)} \rangle$ for all $a, b \in A$ and $h \in H$). If H is finite dimensional then we can take $A = H^*$.

A Hopf algebra is *quasitriangular* [3] if equipped with invertible $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ (summation understood) such that $(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ and $\text{flip} \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}$. Here $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ and so forth. We denote by \bar{H} the quasitriangular Hopf algebra which is the same Hopf algebra as H but with quasitriangular structure $\bar{\mathcal{R}} = \mathcal{R}_{21}^{-1}$. The dual notion, e.g. in [18], of a *coquasitriangular* Hopf algebra A is a Hopf algebra with a convolution-invertible map $\mathcal{R} : A \otimes A \rightarrow k$ satisfying

$$(2.1) \quad \mathcal{R}(ab, c) = \mathcal{R}(a, c_{(1)})\mathcal{R}(b, c_{(2)}), \quad \mathcal{R}(a, bc) = \mathcal{R}(a_{(1)}, c)\mathcal{R}(a_{(2)}, b),$$

$$(2.2) \quad a_{(1)}b_{(1)}\mathcal{R}(b_{(2)}, a_{(2)}) = \mathcal{R}(b_{(1)}, a_{(1)})b_{(2)}a_{(2)}$$

for all $a, b, c \in A$. We define \bar{A} to be the same Hopf algebra as A but with coquasitriangular structure $\bar{\mathcal{R}} = \mathcal{R}_{21}^{-1}$. Equivalently, $\bar{\mathcal{R}}(a, b) = \mathcal{R}(Sb, a)$ for all $a, b \in \bar{A}$. It is shown in [18, 14] that the antipode in this context is invertible.

Let H (resp. A) be (co)quasitriangular. The monoidal categories of left or right (co)modules are braided in the sense of an isomorphism $\Psi_{V, W} : V \otimes W \rightarrow W \otimes V$ obeying axioms similar to the transposition map, but *not* $\Psi_{W, V}\Psi_{V, W} = \text{id}$, given by

$$\begin{aligned} \Psi_L(v \otimes w) &= \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v, & \Psi_R(v \otimes w) &= w \triangleleft \mathcal{R}^{(1)} \otimes v \triangleleft \mathcal{R}^{(2)}, \\ \Psi^L(v \otimes w) &= \mathcal{R}(w^{(\bar{1})}, v^{(\bar{1})})w^{(\bar{\infty})} \otimes v^{(\bar{\infty})}, & \Psi^R(v \otimes w) &= w^{(\bar{0})} \otimes v^{(\bar{0})}\mathcal{R}(v^{(\bar{1})}, w^{(\bar{1})}) \end{aligned}$$

where Ψ_L is the braiding for the left-modules category ${}_H\mathcal{M}$ with action \triangleright , Ψ_R the same for right modules \mathcal{M}_H and action \triangleleft , Ψ^L for the left-comodule category ${}^A\mathcal{M}$ with coaction $\Delta_L v = v^{(\bar{1})} \otimes v^{(\bar{\infty})}$ and Ψ^R for right-comodules \mathcal{M}^A with coaction denoted $\Delta_R v = v^{(\bar{0})} \otimes v^{(\bar{1})}$ (summations understood).

2.2. Bosonisation and cobosonisation. A left H -module algebra B means a Hopf algebra H acting from the left on an algebra B such that $h \triangleright (bc) = (h_{(1)} \triangleright b)(h_{(2)} \triangleright c)$ and $h \triangleright 1 = \epsilon(h)$ for all $b, c \in B$ and $h \in H$. Equivalently, $B \in {}_H\mathcal{M}$ as an algebra, i.e., an object and the product and unit maps are morphisms. One has the familiar smash or cross product algebra which we denote by $B \rtimes H$ built on $B \otimes H$ with $(b \otimes h)(c \otimes g) = b(h_{(1)} \triangleright c) \otimes h_{(2)} g$ for all $b, c \in B$ and $h, g \in H$. Similarly if $B \in \mathcal{M}_H$ as an algebra there is a right cross product algebra $H \ltimes B$ built on $H \otimes B$ with $(h \otimes b)(g \otimes c) = hg_{(1)} \otimes (b \triangleleft g_{(2)})c$. Similarly, given a coalgebra $B \in {}^H\mathcal{M}$ (a left- H -comodule coalgebra or explicitly

$$(2.3) \quad (\text{id} \otimes \Delta)\Delta_L(b) = b_{(1)}^{(\bar{1})} b_{(2)}^{(\bar{1})} \otimes b_{(1)}^{(\bar{\infty})} \otimes b_{(2)}^{(\bar{\infty})}, \quad (\text{id} \otimes \epsilon)\Delta_L = 1 \otimes \epsilon$$

for all $b \in B$) one has a left cross coproduct coalgebra $B \rtimes H$ built on $B \otimes H$ with

$$\Delta(b \otimes h) = b_{(1)} \otimes b_{(2)}^{(\bar{1})} h_{(1)} \otimes b_{(2)}^{(\bar{\infty})} \otimes h_{(2)}.$$

Given a coalgebra $B \in \mathcal{M}^H$ (so a right H -comodule coalgebra or

$$(2.4) \quad (\Delta \otimes \text{id})\Delta_R(b) = b_{(1)}^{(\bar{0})} \otimes b_{(2)}^{(\bar{0})} \otimes b_{(1)}^{(\bar{1})} b_{(2)}^{(\bar{1})}, \quad (\epsilon \otimes \text{id})\Delta_R = \epsilon \otimes 1$$

for all $b \in B$) there is a right cross coproduct coalgebra $H \rtimes B$ built on $H \otimes B$ with

$$\Delta(h \otimes b) = h_{(1)} \otimes b_{(1)}^{(\bar{0})} \otimes h_{(2)} b_{(1)}^{(\bar{1})} \otimes b_{(2)}.$$

We refer to [14] for details. When H is quasitriangular there is a braided monoidal functor ${}_H\mathcal{M} \hookrightarrow {}^H_H\mathcal{M}$ in [9, 14] with a coaction induced by the quasitriangular structure of H so as to form a ‘crossed’ or Yetter-Drinfeld module. Similarly from the right and dually for A coquasitriangular via functors ${}^A\mathcal{M} \hookrightarrow {}^A_A\mathcal{M}$ and ${}^A\mathcal{M} \hookrightarrow \mathcal{M}_A^A$. The latter involve an action induced by the given coaction.

We also need the notion of a ‘braided group’ or Hopf algebra B in a braided category \mathcal{C} , the basic theory of which was worked out in [18, 19, 17]. The unit element is viewed as a morphism $\eta : \underline{1} \rightarrow B$ from the unit object which in our case will just be k . The product, counit, coproduct and antipode are morphisms and we underline the latter two for clarity. In our concrete setting we write $\underline{\Delta}b = b_{(1)} \otimes b_{(2)}$ (summation understood) and recall that $\underline{\Delta}$ is an algebra morphism to the braided tensor product algebra, so that $\underline{\Delta} \cdot = (\cdot \otimes \cdot)(\text{id} \otimes \Psi \otimes \text{id})(\underline{\Delta} \otimes \underline{\Delta})$ with Ψ the braiding on $B \otimes B$. We have [18, 17],

$$(2.5) \quad \underline{S} \circ \cdot = \cdot \circ \Psi \circ (\underline{S} \otimes \underline{S}), \quad \underline{\Delta} \circ \underline{S} = (\underline{S} \otimes \underline{S}) \circ \Psi \circ \underline{\Delta}.$$

Lemma 2.1. [14, 15, 19] *Let H be quasitriangular and $B \in \mathcal{M}_H$ a braided group. Then $H \ltimes B$ by the given action and $H \rtimes B$ by the induced coaction form a Hopf algebra $H \ltimes B$ (the bosonisation of B).*

The coproduct here is $\Delta(h \otimes b) = h_{(1)} \otimes b_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes h_{(2)} \mathcal{R}^{(2)} \otimes b_{(2)}$. Similarly for $B \in {}_H\mathcal{M}$ to give $B \rtimes H$. If A is coquasitriangular and $B \in \mathcal{M}^A$ then the *cobosonisation* is the ordinary Hopf algebra $A \ltimes B$ with $A \rtimes B$ by the given coaction and $A \ltimes B$ by the induced action from the above functor. Explicitly, the cross product is $(a \otimes b)(d \otimes c) = ad_{(1)} \otimes b^{(\bar{0})} c \mathcal{R}(b^{(\bar{1})}, d_{(2)})$. Both constructions are examples of a more general Radford-Majid biproduct theorem [21, 20] (the latter gave the categorical picture) whereby for any Hopf algebra H with invertible antipode, Hopf algebras with split projections to H are of the form $B \rtimes H$ for some $B \in {}^H_H\mathcal{M}$.

Finally, the notion of a dually paired or categorical dual braided group B^* (when B is a rigid object, e.g. finite-dimensional in our applications) in [18, 17] needs a little care to define the pairing $B^* \otimes B^* \otimes B \otimes B$ by pairing $B^* \otimes B$ in the middle first. Pairing maps go to the trivial object. In our context, where objects are built on vector spaces, it is useful to match ordinary Hopf algebra conventions by defining B^* with the adjoint algebra and coalgebra structures in the usual way rather than the above categorical way, which, however, canonically lands B^* in a different category from B .

Lemma 2.2. *Let H be finite dimensional and quasitriangular with dual A , and B be a finite-dimensional braided group in \mathcal{M}_H . Then $B^* \in \mathcal{M}^A$ and $(H \ltimes B)^* = A \ltimes B^*$. Similarly, if $B \in {}_H\mathcal{M}$ then $B^* \in {}^A\mathcal{M}$ and $(B \rtimes H)^* = B^* \rtimes A$.*

2.3. Double bosonisation. Another basic fact about braided groups is that if $B \in \mathcal{C}$ with invertible antipode then B^{op} with the same coalgebra structure as B but with braided-opposite product and antipode given by

$$(2.6) \quad \cdot_{\text{op}} = \cdot \circ \Psi_{B,B}^{-1}, \quad \bar{S} = \underline{S}^{-1}$$

is a braided group in $\bar{\mathcal{C}}$, by which we mean \mathcal{C} with the reversed (inverse) braid crossing [18]. The same remarks apply for $B^{\text{cop}} \in \bar{\mathcal{C}}$ with $\underline{\Delta}_{\text{cop}} = \Psi_{B,B}^{-1} \circ \underline{\Delta}$ and inverted \underline{S} .

If H is quasitriangular and $\mathcal{C} = \mathcal{M}_H$ then $\bar{\mathcal{C}} = \mathcal{M}_{\bar{H}}$. Let B be a braided group in \mathcal{M}_H . By the theory of bosonisation, we have two Hopf algebras $H \bowtie B$ and $B^{*\text{cop}} \bowtie \bar{H}$. We can glue them together to get the following theorem.

Theorem 2.3. *c.f. [10, Theorem 3.2] Let H be a quasitriangular Hopf algebra. Let B be a finite-dimensional braided group in \mathcal{M}_H . There is an ordinary Hopf algebra $B^{*\text{cop}} \bowtie H \bowtie B$, the double bosonisation, built on $B^{*\text{cop}} \otimes H \otimes B$ and containing $B^{*\text{cop}} \bowtie \bar{H}$ and $H \bowtie B$ as sub-Hopf algebras with cross relation*

$$bc = (\mathcal{R}_1^{(2)} \triangleright c_{(2)}) \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} (b_{(2)} \triangleleft \mathcal{R}_2^{-(1)}) (\mathcal{R}_1^{(1)} \triangleright c_{(1)}, b_{(1)} \triangleleft \mathcal{R}_2^{(1)}) \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} c_{(3)}, b_{(3)} \triangleleft \mathcal{R}_2^{-(2)} \rangle.$$

Furthermore, $B^{*\text{cop}} \bowtie H \bowtie B$ has a quasitriangular structure $\mathcal{R}^{\text{new}} = \overline{\text{exp}} \cdot \mathcal{R}$, where $\overline{\text{exp}} = \sum f^a \otimes \underline{S} e_a$, $\{e_a\}$ is a basis of B and $\{f^a\}$ is a dual basis of B^* .

In fact, B in [10] is not required to be finite dimensional but we have restricted to the finite-dimensional case for simplicity. Our goal is a dual version of this theorem with A coquasitriangular and $B \in {}^A\mathcal{M}$, in which case the category with reversed braiding is ${}^A\mathcal{M}$ and

$$(2.7) \quad a \cdot_{\text{op}} b = \mathcal{R}(Sa^{(\bar{1})}, b^{(\bar{1})}) b^{(\bar{\infty})} a^{(\bar{\infty})}$$

for all $a, b \in B^{\text{op}}$. As in Lemma 2.2, we think of ${}^A\mathcal{M}$ as \mathcal{M}_H in the finite-dimensional Hopf algebra case by evaluating against a coaction of A to get an action of H .

Lemma 2.4. *If H is finite dimensional and quasitriangular with dual A and $B \in {}^A\mathcal{M}$ is finite dimensional then $(B^{\text{op}})^* = B^{*\text{cop}} \in {}_{\bar{H}}\mathcal{M}$.*

Proof. Here $B^{\text{op}} \in {}^{\bar{A}}\mathcal{M}$ or $\mathcal{M}_{\bar{H}}$ and $(B^{\text{op}})^* \in {}_{\bar{H}}\mathcal{M}$ where $B^{*\text{cop}}$ lives. It is clear that the coproduct of B^{op} corresponds to the product of $B^{*\text{cop}}$. For the other half,

$$\begin{aligned} \langle x, b \cdot_{\text{op}} c \rangle &= \langle x, c^{(\bar{\infty})} b^{(\bar{\infty})} \rangle \mathcal{R}(Sb^{(\bar{1})}, c^{(\bar{1})}) = \langle x_{(1)}, c^{(\bar{\infty})} \rangle \langle x_{(2)}, b^{(\bar{\infty})} \rangle \langle b^{(\bar{1})}, \mathcal{R}^{-(2)} \rangle \langle c^{(\bar{1})}, \mathcal{R}^{-(1)} \rangle \\ &= \langle x_{(1)}, c \triangleleft \mathcal{R}^{-(1)} \rangle \langle x_{(2)}, b \triangleleft \mathcal{R}^{-(2)} \rangle = \langle \mathcal{R}^{-(2)} \triangleright x_{(2)} \otimes \mathcal{R}^{-(1)} \triangleright x_{(1)}, b \otimes c \rangle \end{aligned}$$

which is $\langle \underline{\Delta}_{\text{cop}} x, b \otimes c \rangle$ as required. \square

3. CO-DOUBLE BOSONISATION

The dual version of Theorem 2.3 can in principle now be deduced using the lemmas in the preceding section, at least when A is finite dimensional. However, we do not want to be limited to this case and give a direct proof of the resulting formulae.

Theorem 3.1 (Co-double bosonisation). *Let B be a finite-dimensional braided group in ${}^A\mathcal{M}$ with basis $\{e_a\}$. Denote its dual by $B^* \in \mathcal{M}^A$ with dual basis $\{f^a\}$. Then there is an ordinary Hopf algebra $B^{\text{op}} \bowtie A \bowtie B^*$, the co-double bosonisation, built on the vector space $B^{\text{op}} \otimes A \otimes B^*$ with*

$$\begin{aligned} (x \otimes k \otimes y)(w \otimes \ell \otimes z) &= x \cdot_{\text{op}} w^{(\bar{\infty})} \otimes k_{(2)} \ell_{(1)} \otimes y^{(\bar{0})} z \mathcal{R}(y^{(\bar{1})}, \ell_{(2)}) \mathcal{R}(Sk_{(1)}, w^{(\bar{1})}), \\ \Delta(x \otimes k \otimes y) &= \sum_a x_{(1)} \otimes x_{(2)}^{(\bar{1})} k_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{a(3)}^{(\bar{\infty})} \otimes k_{(4)} y_{(1)}^{(\bar{1})} \otimes y_{(2)} \\ &\quad \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} k_{(2)}) \mathcal{R}(S(k_{(3)} y_{(1)}^{(\bar{1})} k_{(1)}), e_{a(3)}^{(\bar{1})}) \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \end{aligned}$$

for all $x, w \in B^{\text{op}}$, $k, \ell \in A$, and $y, z \in B^*$.

Here B^{op} , A and B^* are subalgebras of $B^{\text{op}} \bowtie A \ltimes B^*$ and identifying $x = x \otimes 1 \otimes 1$, $k = 1 \otimes k \otimes 1$ and $y = 1 \otimes 1 \otimes y$ we have $xky \equiv x \otimes k \otimes y$. We also have algebra maps

$$B^* \hookrightarrow B^{\text{op}} \bowtie A \ltimes B^* \twoheadrightarrow B^{\text{op}} \bowtie A, \quad B^{\text{op}} \hookrightarrow B^{\text{op}} \bowtie A \ltimes B^* \twoheadrightarrow A \ltimes B^*$$

where the surjections are $\text{id} \otimes \epsilon$ and $\epsilon \otimes \text{id}$ respectively. It remains to prove Theorem 3.1.

Lemma 3.2. *The product stated in Theorem 3.1 is associative.*

Proof. We expand the definition of the product to find

$$\begin{aligned} & \left((x \otimes k \otimes y)(w \otimes \ell \otimes z) \right) (m \otimes j \otimes v) \\ &= (x \cdot_{\text{op}} w^{(\infty)} \otimes k_{(2)} \ell_{(1)} \otimes y^{(\bar{0})} z) (m \otimes j \otimes v) \mathcal{R}(y^{(\bar{1})}, \ell_{(2)}) \mathcal{R}(Sk_{(1)}, w^{(\bar{1})}) \\ &= x \cdot_{\text{op}} w^{(\infty)} \cdot_{\text{op}} m^{(\infty)} \otimes k_{(3)} \ell_{(2)} j_{(1)} \otimes y^{(\bar{0})(\bar{0})} z^{(\bar{0})} v \mathcal{R}(y^{(\bar{1})}, \ell_{(3)}) \mathcal{R}(Sk_{(1)}, w^{(\bar{1})}) \\ & \quad \mathcal{R}(y^{(\bar{0})(\bar{1})} z^{(\bar{1})}, j_{(2)}) \mathcal{R}(S(k_{(2)} \ell_{(1)}), m^{(\bar{1})}) \\ &= x \cdot_{\text{op}} w^{(\infty)} \cdot_{\text{op}} m^{(\infty)} \otimes k_{(3)} \ell_{(2)} j_{(1)} \otimes y^{(\bar{0})} z^{(\bar{0})} v \mathcal{R}(y^{(\bar{1})}_{(2)}, \ell_{(3)}) \mathcal{R}(Sk_{(1)}, w^{(\bar{1})}) \\ & \quad \mathcal{R}(y^{(\bar{1})}_{(1)}, j_{(2)}) \mathcal{R}(z^{(\bar{1})}, j_{(3)}) \mathcal{R}(S\ell_{(1)}, m^{(\bar{1})}_{(1)}) \mathcal{R}(Sk_{(2)}, m^{(\bar{1})}_{(2)}), \end{aligned}$$

where the last equality uses the right-coaction property on y . Similarly,

$$\begin{aligned} & (x \otimes k \otimes y) \left((w \otimes \ell \otimes z)(m \otimes j \otimes v) \right) \\ &= (x \otimes k \otimes y) (w \cdot_{\text{op}} m^{(\infty)} \otimes \ell_{(2)} j_{(1)} \otimes z^{(\bar{0})} v) \mathcal{R}(z^{(\bar{1})}, j_{(2)}) \mathcal{R}(S\ell_{(1)}, m^{(\bar{1})}) \\ &= x \cdot_{\text{op}} w^{(\infty)} \cdot_{\text{op}} m^{(\infty)(\infty)} \otimes k_{(2)} \ell_{(2)} j_{(1)} \otimes y^{(\bar{0})} z^{(\bar{0})} v \mathcal{R}(z^{(\bar{1})}, j_{(3)}) \mathcal{R}(S\ell_{(1)}, m^{(\bar{1})}) \\ & \quad \mathcal{R}(y^{(\bar{1})}, \ell_{(3)} j_{(2)}) \mathcal{R}(Sk_{(1)}, w^{(1)} m^{(\infty)(\bar{1})}), \end{aligned}$$

which by the left-coaction property on m agrees with our first calculation. \square

Lemma 3.3. *The coproduct Δ stated in Theorem 3.1 is an algebra map.*

Proof. Expanding the product and then the coproduct, we have

$$\begin{aligned} & \Delta \left((x \otimes k \otimes y)(w \otimes \ell \otimes z) \right) \\ &= x_{(1)} \cdot_{\text{op}} w^{(\infty)}_{(1)} \otimes x_{(2)} \otimes w^{(\infty)}_{(2)} \otimes k_{(2)} \ell_{(1)} \otimes f^a \\ & \quad \otimes e_{a(1)} \cdot_{\text{op}} x_{(2)} \cdot_{\text{op}} w^{(\infty)}_{(2)} \cdot_{\text{op}} \bar{S}e_{a(3)} \otimes k_{(5)} \ell_{(4)} y^{(\bar{0})}_{(1)} z_{(2)} z_{(1)}^{(\bar{0})(\bar{1})} \\ & \quad \otimes y^{(\bar{0})}_{(2)} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)} w^{(\infty)}_{(2)} \otimes k_{(3)} \ell_{(2)}) \mathcal{R}(S(k_{(4)} \ell_{(3)} y^{(\bar{0})}_{(1)} z_{(1)}^{(\bar{0})(\bar{1})}), e_{a(3)}^{(\bar{1})}) \\ & \quad \mathcal{R}(Sk_{(1)}, w^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(5)}) \mathcal{R}(Sx_{(2)}^{(\bar{1})}, w^{(\infty)}_{(1)} \otimes k_{(2)} \ell_{(1)}) \mathcal{R}(y^{(\bar{0})}_{(2)} z_{(2)}^{(\bar{1})}, z_{(1)}^{(\bar{1})}) \langle y^{(\bar{0})}_{(1)} z_{(1)}^{(\bar{0})(\bar{0})}, e_{a(2)} \rangle \\ &= x_{(1)} \cdot_{\text{op}} w_{(1)} \otimes x_{(2)} \otimes w_{(2)} \otimes k_{(2)} \ell_{(1)} \otimes f^a \otimes e_{a(1)} \cdot_{\text{op}} x_{(2)} \cdot_{\text{op}} w_{(2)} \cdot_{\text{op}} \bar{S}e_{a(4)} \\ & \quad \otimes k_{(5)} \ell_{(4)} y_{(1)} z_{(1)} \otimes y_{(2)} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)} w_{(2)} \otimes k_{(3)} \ell_{(2)}) \\ & \quad \mathcal{R}(S(k_{(4)} \ell_{(3)} y_{(1)} z_{(1)}), e_{a(4)}^{(\bar{1})}) \mathcal{R}(Sk_{(1)}, w_{(1)} \otimes w_{(2)} \otimes y_{(3)} y_{(2)} \otimes \ell_{(5)}) \\ & \quad \mathcal{R}(Sx_{(2)}^{(\bar{1})}, w_{(1)} \otimes w_{(2)} \otimes y_{(3)} y_{(2)} \otimes \ell_{(5)}) \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{a(3)} \rangle \\ &= x_{(1)} \cdot_{\text{op}} w_{(1)} \otimes x_{(2)} \otimes w_{(2)} \otimes k_{(3)} \ell_{(2)} \otimes f^a \otimes e_{a(1)} \cdot_{\text{op}} x_{(2)} \cdot_{\text{op}} w_{(2)} \cdot_{\text{op}} \bar{S}e_{a(4)} \\ & \quad \otimes k_{(6)} \ell_{(4)} y_{(1)} z_{(1)} \otimes y_{(2)} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)} w_{(2)} \otimes k_{(4)} \ell_{(2)}) \\ & \quad \mathcal{R}(S(k_{(5)} \ell_{(3)} y_{(1)} z_{(1)}), e_{a(4)}^{(\bar{1})}) \mathcal{R}(Sx_{(2)}^{(\bar{1})}, k_{(1)}) \otimes w_{(1)} \mathcal{R}(Sk_{(2)}, w_{(2)} \otimes y_{(3)} y_{(2)} \otimes \ell_{(5)}) \\ & \quad \mathcal{R}(y_{(1)}^{(\bar{1})}, \ell_{(5)}) \mathcal{R}(y_{(2)}^{(\bar{1})}, \ell_{(6)} z_{(1)}) \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{a(3)} \rangle \end{aligned}$$

for all $x, w \in B^{\text{op}}$, $k, \ell \in A$, $y, z \in B^*$. The second equality uses the comodule coalgebra property (2.3) on w and coassociativity. The last expression uses coquasitriangularity (2.1) to gather the parts of $w_{(1)}^{(\bar{1})}$ and $y_{(2)}^{(\bar{1})}$ inside \mathcal{R} . On the other side,

$$\begin{aligned} & \Delta(x \otimes k \otimes y) \Delta(w \otimes \ell \otimes z) \\ &= x_{(1)} \cdot_{\text{op}} w_{(1)} \otimes x_{(2)} \otimes k_{(1)} w_{(2)} \otimes \ell_{(1)} \otimes f^a f^b \\ & \quad \otimes e_{a(1)} \cdot_{\text{op}} x_{(2)} \cdot_{\text{op}} \bar{S}e_{a(3)} \cdot_{\text{op}} e_{b(1)} \cdot_{\text{op}} w_{(2)} \cdot_{\text{op}} \bar{S}e_{b(3)} \\ & \quad \otimes k_{(4)} y_{(1)} \otimes \ell_{(4)} z_{(1)} \otimes y_{(2)} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)} \otimes k_{(2)}) \end{aligned}$$

$$\begin{aligned}
& \mathcal{R}(S(k_{(3)}y_{(1)}^{(\bar{1})}_{(1)}), e_{a(3)}^{(1)}) \mathcal{R}(e_{b(1)}^{(\bar{1})}, w_{(2)}^{(\bar{1})}_{(2)} \ell_{(2)}) \mathcal{R}(S(\ell_{(3)}z_{(1)}^{(\bar{1})}_{(1)}), e_{b(3)}^{(\bar{1})}) \\
& \mathcal{R}(S(x_{(2)}^{(\bar{1})}_{(1)} k_{(1)}^{(1)}), w_{(1)}^{(\bar{1})}) \mathcal{R}(f^{a(\bar{1})}, w_{(2)}^{(\bar{1})}_{(1)} \ell_{(1)(2)}) \\
& \mathcal{R}(S(k_{(4)}z_{(1)}^{(\bar{1})}_{(2)}), e_{b(1)}^{(\bar{1})} w_{(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})}) \mathcal{R}(y_{(2)}^{(\bar{1})}, \ell_{(4)(2)} z_{(1)}^{(\bar{1})}_{(2)}) \\
& \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{b(2)} \rangle \\
& = x_{(1)} \cdot_{\text{op}} w_{(1)}^{(\bar{\infty})} \otimes x_{(2)}^{(\bar{1})}_{(2)} k_{(2)} w_{(2)}^{(\bar{1})}_{(1)} \ell_{(1)} \otimes f^a f^b \\
& \quad \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(3)}^{(\bar{\infty})} \cdot_{\text{op}} e_{a(4)}^{(\bar{\infty})} \cdot_{\text{op}} w_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(6)}^{(\bar{\infty})} \\
& \quad \otimes k_{(6)} y_{(1)}^{(\bar{1})}_{(3)} \ell_{(5)} z_{(1)}^{(\bar{1})}_{(2)} \otimes y_{(2)}^{(\bar{0})} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{\infty})}, x_{(2)}^{(\bar{1})}_{(3)} k_{(3)}) \\
& \mathcal{R}(S(k_{(4)}y_{(1)}^{(\bar{1})}_{(1)}), e_{a(3)}^{(\bar{\infty})}) \mathcal{R}(e_{a(4)}^{(\bar{1})}, w_{(2)}^{(\bar{1})}_{(3)} \ell_{(3)}) \mathcal{R}(S(\ell_{(4)}z_{(1)}^{(\bar{1})}_{(1)}), e_{a(6)}^{(\bar{1})}) \\
& \mathcal{R}(S(x_{(2)}^{(\bar{1})}_{(1)} k_{(1)}), w_{(1)}^{(\bar{1})}) \mathcal{R}(e_{a(1)}^{(\bar{1})} e_{a(2)}^{(\bar{1})} e_{a(3)}^{(\bar{1})}, w_{(2)}^{(\bar{1})}_{(2)} \ell_{(2)}) \\
& \mathcal{R}(S(k_{(5)}y_{(1)}^{(\bar{1})}_{(2)}), e_{a(4)}^{(\bar{\infty})} w_{(2)}^{(\bar{\infty})} e_{a(6)}^{(\bar{\infty})}) \mathcal{R}(y_{(2)}^{(\bar{1})}, \ell_{(6)} z_{(1)}^{(\bar{1})}_{(3)}) \\
& \langle y_{(1)}^{(\bar{0})}, e_{a(2)}^{(\bar{\infty})} \rangle \langle z_{(1)}^{(\bar{0})}, e_{a(5)} \rangle \\
& = x_{(1)} \cdot_{\text{op}} w_{(1)}^{(\bar{\infty})} \otimes x_{(2)}^{(\bar{1})}_{(2)} k_{(2)} w_{(2)}^{(\bar{1})}_{(1)} \ell_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} w_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(4)}^{(\bar{\infty})} \\
& \quad \otimes k_{(5)} y_{(1)}^{(\bar{1})}_{(3)} \ell_{(4)} z_{(1)}^{(\bar{1})}_{(2)} \otimes y_{(2)}^{(\bar{0})} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})}_{(3)} k_{(3)}) \mathcal{R}(S(\ell_{(3)}z_{(1)}^{(\bar{1})}_{(1)}), e_{a(4)}^{(\bar{1})}_{(1)}) \\
& \mathcal{R}(S(x_{(2)}^{(\bar{1})}_{(1)} k_{(1)}), w_{(1)}^{(\bar{1})}) \mathcal{R}(e_{a(1)}^{(\bar{1})} y_{(1)}^{(\bar{1})}_{(1)} w_{(2)}^{(\bar{1})}_{(2)} \ell_{(2)}) \\
& \mathcal{R}(S(k_{(4)}y_{(1)}^{(\bar{1})}_{(2)}), w_{(2)}^{(\bar{1})}_{(3)} e_{a(4)}^{(\bar{1})}_{(2)}) \mathcal{R}(y_{(2)}^{(\bar{1})}, \ell_{(5)} z_{(1)}^{(\bar{1})}_{(3)}) \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{a(3)} \rangle \\
& = x_{(1)} \cdot_{\text{op}} w_{(1)}^{(\bar{\infty})} \otimes x_{(2)}^{(\bar{1})}_{(2)} k_{(2)} w_{(2)}^{(\bar{1})}_{(1)} \ell_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} w_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(4)}^{(\bar{\infty})} \\
& \quad \otimes k_{(6)} y_{(1)}^{(\bar{1})}_{(3)} \ell_{(5)} z_{(1)}^{(\bar{1})}_{(2)} \otimes y_{(2)}^{(\bar{0})} z_{(2)} \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})}_{(3)} k_{(3)} w_{(2)}^{(\bar{1})}_{(2)} \ell_{(2)}) \\
& \mathcal{R}(S(k_{(5)}y_{(1)}^{(\bar{1})}_{(2)} \ell_{(4)} z_{(1)}^{(\bar{1})}_{(1)}), e_{a(4)}^{(\bar{1})}) \mathcal{R}(S(x_{(2)}^{(\bar{1})}_{(1)} k_{(1)}), w_{(1)}^{(\bar{1})}) \mathcal{R}(y_{(2)}^{(\bar{1})}, \ell_{(6)} z_{(1)}^{(\bar{1})}_{(3)}) \\
& \mathcal{R}(y_{(1)}^{(\bar{1})}_{(1)}, \ell_{(3)}) \mathcal{R}(S k_{(4)}, w_{(2)}^{(\bar{1})}_{(3)}) \langle y_{(1)}^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{a(3)} \rangle,
\end{aligned}$$

where the second equality uses duality $\langle f^{a(\bar{0})}, e_a \rangle f^{a(\bar{1})} = \langle f^a, e_a^{(\bar{\infty})} \rangle e_a^{(\bar{1})}$ followed by the comodule coalgebra property (2.3) on e_a . The third equality cancels $(\bar{S}e_{a(3)} \cdot_{\text{op}} e_{a(4)})^{(\bar{\infty})}$ making all subsequent coactions trivial. The fourth equality uses (2.1) to gather the parts of $e_{a(1)}^{(\bar{1})}$ and $e_{a(4)}^{(\bar{1})}$ inside \mathcal{R} , and cancels some \mathcal{R} s. In the final expression, one can use quasicommutativity (2.2) to reorder the second tensor factor so as to coincide with the result of the first calculation. \square

Lemma 3.4. *The coproduct Δ stated in Theorem 3.1 is coassociative.*

Proof. We expand the definition of the coproduct to find

$$\begin{aligned}
& (\text{id} \otimes \Delta) \Delta(x \otimes k \otimes y) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})}_{(1)} x_{(3)}^{(\bar{1})}_{(1)} k_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(5)}^{(\bar{\infty})} \\
& \quad \otimes e_{a(2)}^{(\bar{1})}_{(4)} x_{(3)}^{(\bar{1})}_{(4)} e_{a(4)}^{(\bar{1})}_{(3)} k_{(4)} y_{(1)}^{(\bar{1})}_{(2)} \otimes f^b \\
& \quad \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} e_{a(2)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(4)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{b(3)}^{(\bar{\infty})} \otimes k_{(7)} y_{(1)}^{(\bar{1})}_{(5)} y_{(2)}^{(\bar{1})}_{(2)} \otimes y_{(3)} \\
& \quad \langle y_{(1)}^{(\bar{0})}, e_{a(3)} \rangle \langle y_{(2)}^{(\bar{0})}, e_{b(2)} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, e_{a(2)}^{(\bar{1})}_{(5)} x_{(3)}^{(\bar{1})}_{(5)} e_{a(4)}^{(\bar{1})}_{(4)} k_{(5)} y_{(1)}^{(\bar{1})}_{(3)}) \\
& \mathcal{R}(S(k_{(6)}y_{(1)}^{(\bar{1})}_{(4)} y_{(2)}^{(\bar{1})}_{(1)}), e_{b(3)}^{(\bar{1})}) \mathcal{R}(e_{a(1)}^{(\bar{1})} e_{a(2)}^{(\bar{1})}_{(1)} x_{(2)}^{(\bar{1})}_{(2)} x_{(3)}^{(\bar{1})}_{(2)} k_{(2)}) \\
& \mathcal{R}(S(k_{(3)}y_{(1)}^{(\bar{1})}_{(1)}), e_{a(4)}^{(\bar{1})}_{(1)} e_{a(5)}^{(\bar{1})}_{(1)}) \mathcal{R}(S(e_{a(2)}^{(\bar{1})}_{(3)} x_{(3)}^{(\bar{1})}_{(3)}), e_{a(5)}^{(\bar{1})}_{(3)}) \\
& \mathcal{R}(S e_{a(4)}^{(\bar{1})}_{(2)}, e_{a(5)}^{(\bar{1})}_{(2)}) \mathcal{R}(S e_{a(2)}^{(\bar{1})}_{(2)}, x_{(2)}^{(\bar{1})}_{(3)}) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})}_{(1)} x_{(3)}^{(\bar{1})}_{(1)} k_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(5)}^{(\bar{\infty})} \\
& \quad \otimes e_{a(2)}^{(\bar{1})}_{(3)} x_{(3)}^{(\bar{1})}_{(5)} e_{a(4)}^{(\bar{1})}_{(3)} k_{(6)} y_{(1)}^{(\bar{1})}_{(3)} \otimes f^b \\
& \quad \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} e_{a(2)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{a(4)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S}e_{b(3)}^{(\bar{\infty})} \otimes k_{(9)} y_{(1)}^{(\bar{1})}_{(6)} y_{(2)}^{(\bar{1})}_{(2)} \otimes y_{(3)} \\
& \quad \langle y_{(1)}^{(\bar{0})}, e_{a(3)} \rangle \langle y_{(2)}^{(\bar{0})}, e_{b(2)} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, e_{a(2)}^{(\bar{1})}_{(4)} x_{(3)}^{(\bar{1})}_{(6)} e_{a(4)}^{(\bar{1})}_{(4)} k_{(7)} y_{(1)}^{(\bar{1})}_{(4)}) \\
& \mathcal{R}(S(k_{(8)}y_{(1)}^{(\bar{1})}_{(5)} y_{(2)}^{(\bar{1})}_{(1)}), e_{b(3)}^{(\bar{1})}) \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})}_{(2)} x_{(3)}^{(\bar{1})}_{(2)} k_{(2)})
\end{aligned}$$

$$\begin{aligned}
& \mathcal{R}(e_{a(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(3)}) \mathcal{R}(S(e_{a(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} e_{a(4)}^{(\bar{1})} k_{(5)} y_{(1)}^{(\bar{1})}), e_{a(5)}^{(\bar{1})}) \\
& \mathcal{R}(S(k_{(4)} y_{(1)}^{(\bar{1})}), e_{a(4)}^{(\bar{1})}) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{a(5)}^{(\bar{\infty})} \\
& \quad \otimes x_{(3)}^{(\bar{1})} e_{a(2)}^{(\bar{1})} k_{(5)} e_{a(4)}^{(\bar{1})} y_{(1)}^{(\bar{1})} \otimes f^b \\
& \quad \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} e_{a(2)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{a(4)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{b(3)}^{(\bar{\infty})} \otimes k_{(9)} y_{(1)}^{(\bar{1})} y_{(2)}^{(\bar{1})} \otimes y_{(3)} \\
& \quad \langle y_{(1)}^{(\bar{0})}, e_{a(3)}^{(\bar{0})} \rangle \langle y_{(2)}^{(\bar{0})}, e_{b(2)}^{(\bar{0})} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(3)}^{(\bar{1})} e_{a(2)}^{(\bar{1})} k_{(6)} e_{a(4)}^{(\bar{1})} y_{(1)}^{(\bar{1})}) \\
& \quad \mathcal{R}(S(k_{(8)} y_{(1)}^{(\bar{1})} y_{(2)}^{(\bar{1})}), e_{b(3)}^{(\bar{1})}) \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \\
& \quad \mathcal{R}(S(x_{(3)}^{(\bar{1})} e_{a(2)}^{(\bar{1})} k_{(4)} e_{a(4)}^{(\bar{1})} y_{(1)}^{(\bar{1})}), e_{a(5)}^{(\bar{1})}) \mathcal{R}(e_{a(2)}^{(\bar{1})}, k_{(3)}) \\
& \quad \mathcal{R}(e_{a(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(5)}), x_{(3)}^{(\bar{1})} k_{(6)}) \mathcal{R}(S y_{(1)}^{(\bar{1})}, e_{a(4)}^{(\bar{1})}) \mathcal{R}(S k_{(7)}, e_{a(4)}^{(\bar{1})}),
\end{aligned}$$

where the second equality uses (2.1) to gather the parts of $e_{a(1)}^{(\bar{1})}$ and $e_{a(5)}^{(\bar{1})}$, cancelling some of the \mathcal{R} s. We lastly use (2.2) to change the order in the fifth tensor factor and in a similar term inside \mathcal{R} , again cancelling some of the \mathcal{R} s. On the other side,

$$\begin{aligned}
& (\Delta \otimes \text{id}) \Delta(x \otimes k \otimes y) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^b \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{b(3)}^{(\bar{\infty})} \otimes x_{(3)}^{(\bar{1})} k_{(4)} f_{(1)}^{a(\bar{1})} \\
& \quad \otimes f_{(2)}^a \otimes e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{a(3)}^{(\bar{\infty})} \otimes k_{(7)} y_{(1)}^{(\bar{1})} \otimes y_{(2)} \langle y_{(1)}^{(\bar{0})}, e_{a(2)}^{(\bar{0})} \rangle \langle f_{(1)}^{a(\bar{0})}, e_{b(2)}^{(\bar{0})} \rangle \\
& \quad \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \mathcal{R}(S(x_{(3)}^{(\bar{1})} k_{(3)} f_{(1)}^{a(\bar{1})}), e_{b(3)}^{(\bar{1})}) \\
& \quad \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(3)}^{(\bar{1})} k_{(5)}) \mathcal{R}(S(k_{(6)} y_{(1)}^{(\bar{1})}), e_{a(3)}^{(\bar{1})}) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^b \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{b(3)}^{(\bar{\infty})} \otimes x_{(3)}^{(\bar{1})} k_{(4)} e_{b(2)}^{(\bar{1})} \\
& \quad \otimes f^c \otimes (e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})})_{(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \bar{S} (e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})})_{(3)}^{(\bar{\infty})} \otimes k_{(7)} y_{(1)}^{(\bar{1})} \otimes y_{(2)} \\
& \quad \langle f^a, e_{b(2)}^{(\bar{\infty})} \rangle \langle y_{(1)}^{(\bar{0})}, (e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})})_{(2)}^{(\bar{\infty})} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \\
& \quad \mathcal{R}(S(x_{(3)}^{(\bar{1})} k_{(3)} e_{b(2)}^{(\bar{1})}), e_{b(3)}^{(\bar{1})}) \mathcal{R}((e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})})_{(1)}^{(\bar{1})}, x_{(3)}^{(\bar{1})} k_{(5)}) \\
& \quad \mathcal{R}(S(k_{(6)} y_{(1)}^{(\bar{1})}), (e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})})_{(3)}^{(\bar{1})}) \mathcal{R}(e_c^{(\bar{1})}, e_a^{(\bar{1})}) \\
& = x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^b \otimes e_{b(1)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(2)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{b(5)}^{(\bar{\infty})} \\
& \quad \otimes x_{(3)}^{(\bar{1})} k_{(4)} e_{b(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})} \otimes f^c \\
& \quad \otimes e_{c(1)}^{(\bar{\infty})} \cdot_{\text{op}} e_{b(2)}^{(\bar{\infty})} \cdot_{\text{op}} x_{(3)}^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} (e_{c(3)}^{(\bar{\infty})} \cdot_{\text{op}} e_{b(4)}^{(\bar{\infty})}) \otimes k_{(7)} y_{(1)}^{(\bar{1})} \otimes y_{(2)} \\
& \quad \langle y_{(1)}^{(\bar{0})}, e_{c(2)}^{(\bar{\infty})} \cdot_{\text{op}} e_{b(3)}^{(\bar{\infty})} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \\
& \quad \mathcal{R}(S(x_{(3)}^{(\bar{1})} k_{(3)} e_{b(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}), e_{b(5)}^{(\bar{1})}) \mathcal{R}(e_{c(1)}^{(\bar{1})}, e_{b(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})} k_{(5)}) \\
& \quad \mathcal{R}(S(k_{(6)} y_{(1)}^{(\bar{1})}), e_{c(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}) \mathcal{R}(e_{c(1)}^{(\bar{1})} e_{c(2)}^{(\bar{1})} e_{c(3)}^{(\bar{1})}, e_{b(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}) \\
& \quad \mathcal{R}(S e_{c(3)}^{(\bar{1})}, e_{b(2)}^{(\bar{1})} e_{b(3)}^{(\bar{1})}) \mathcal{R}(S e_{c(2)}^{(\bar{1})}, e_{b(2)}^{(\bar{1})}).
\end{aligned}$$

For the second equality we use duality $\langle f_{(1)}^{a(\bar{0})}, e_{b(2)}^{(\bar{0})} \rangle f_{(1)}^{a(\bar{1})} = \langle f_{(1)}^{a(\bar{1})}, e_{b(2)}^{(\bar{\infty})} \rangle e_{b(2)}^{(\bar{1})}$ to replace $f_{(1)}^{a(\bar{1})}$ by $e_{b(2)}^{(\bar{1})}$, followed by

$$e_a \otimes f_{(1)}^a \otimes f_{(2)}^a = e_c^{(\bar{\infty})} \cdot_{\text{op}} e_a^{(\bar{\infty})} \otimes f^a \otimes f^c \mathcal{R}(e_c^{(\bar{1})}, e_a^{(\bar{1})})$$

to replace $f_{(1)}^a \otimes f_{(2)}^a$ by $f^a \otimes f^c$. For the third equality, we use $\langle f^a, e_{b(2)}^{(\bar{\infty})} \rangle$ to replace e_a by $e_{b(2)}^{(\bar{\infty})}$, after which we expand $(e_c^{(\bar{\infty})} \cdot_{\text{op}} e_{b(2)}^{(\bar{\infty})})_{(1)}^{(\bar{\infty})}$ etc. using $\underline{\Delta}$ a braided-homomorphism. In the last expression, we expand \bar{S} of a \cdot_{op} product and use $\langle y_{(1)}^{(\bar{0})}, e_{c(2)}^{(\bar{\infty})} \cdot_{\text{op}} e_{b(3)}^{(\bar{\infty})} \rangle = \langle y_{(1)}^{(\bar{0})}, e_{b(3)}^{(\bar{\infty})} \rangle \langle y_{(1)}^{(\bar{0})}, e_{c(2)}^{(\bar{\infty})} \rangle \mathcal{R}(S e_{c(2)}^{(\bar{\infty})}, e_{b(3)}^{(\bar{\infty})})$.

By the comodule coalgebra property (2.4), the first pairing on the right becomes $\langle y_{(1)}^{(\bar{0})}, e_{b(3)}^{(\bar{\infty})} \rangle \langle y_{(2)}^{(\bar{0})}, e_{c(2)}^{(\bar{\infty})} \rangle$ and duality $\langle y_{(1)}^{(\bar{0})}, e_{b(3)}^{(\bar{\infty})} \rangle e_{b(3)}^{(\bar{1})} = \langle y_{(1)}^{(\bar{0})}, e_{b(3)}^{(\bar{0})} \rangle y_{(1)}^{(\bar{0})}$ replaces $e_{b(3)}^{(\bar{1})}$ by $y_{(1)}^{(\bar{0})}$. The other pairing similarly replaces $e_{c(2)}^{(\bar{1})}$ by $y_{(2)}^{(\bar{0})}$, so

$$(\Delta \otimes \text{id}) \Delta(x \otimes k \otimes y)$$

$$\begin{aligned}
&= x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^b \otimes e_{b(1)}^{(\infty)} \cdot_{\text{op}} x_{(2)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(5)}^{(\infty)} \\
&\quad \otimes x_{(3)}^{(\bar{1})} k_{(4)} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})} \otimes f^c \\
&\quad \otimes e_{c(1)}^{(\infty)} \cdot_{\text{op}} e_{b(2)}^{(\infty)} \cdot_{\text{op}} x_{(3)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(4)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{c(3)}^{(\infty)} \otimes k_{(7)} y_{(1)}^{(\bar{1})} y_{(2)}^{(\bar{1})} \otimes y_{(3)} \\
&\quad \langle y_{(1)}^{(\bar{0})}, e_{b(3)} \rangle \langle y_{(2)}^{(\bar{0})}, e_{c(2)} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \\
&\quad \mathcal{R}(S(x_{(3)}^{(\bar{1})} k_{(3)} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}), e_{b(5)}^{(\bar{1})}) \mathcal{R}(e_{c(1)}^{(\bar{1})} e_{b(2)}^{(\bar{1})} e_{c(2)}^{(\bar{1})} e_{b(4)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(3)}^{(\bar{1})}) \\
&\quad \mathcal{R}(S(e_{c(3)}^{(\bar{1})} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})}) \mathcal{R}(S y_{(2)}^{(\bar{1})} e_{b(2)}^{(\bar{1})} \mathcal{R}(S y_{(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})}) \mathcal{R}(S e_{c(3)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}) \\
&= x_{(1)} \otimes x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(1)} \otimes f^b \otimes e_{b(1)}^{(\infty)} \cdot_{\text{op}} x_{(2)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(5)}^{(\infty)} \\
&\quad \otimes x_{(3)}^{(\bar{1})} k_{(4)} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})} \otimes f^c \\
&\quad \otimes e_{c(1)}^{(\infty)} \cdot_{\text{op}} e_{b(2)}^{(\infty)} \cdot_{\text{op}} x_{(3)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(4)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{c(3)}^{(\infty)} \otimes k_{(9)} y_{(1)}^{(\bar{1})} y_{(2)}^{(\bar{1})} \otimes y_{(3)} \\
&\quad \langle y_{(1)}^{(\bar{0})}, e_{b(3)} \rangle \langle y_{(2)}^{(\bar{0})}, e_{c(2)} \rangle \mathcal{R}(e_{b(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} x_{(3)}^{(\bar{1})} k_{(2)}) \\
&\quad \mathcal{R}(S(x_{(3)}^{(\bar{1})} k_{(3)} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}), e_{b(5)}^{(\bar{1})}) \mathcal{R}(S(k_{(8)} y_{(1)}^{(\bar{1})} y_{(2)}^{(\bar{1})}), e_{c(3)}^{(\bar{1})}) \\
&\quad \mathcal{R}(S(k_{(7)} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}) \mathcal{R}(e_{c(1)}^{(\bar{1})}, x_{(3)}^{(\bar{1})} k_{(5)} e_{b(2)}^{(\bar{1})} y_{(1)}^{(\bar{1})} e_{b(4)}^{(\bar{1})}) \\
&\quad \mathcal{R}(e_{b(2)}^{(\bar{1})}, x_{(3)}^{(\bar{1})} k_{(6)}),
\end{aligned}$$

using (2.1) to gather $e_{c(1)}^{(\bar{1})}$ and $e_{c(3)}^{(\bar{1})}$, and cancelling some \mathcal{R} s. In the final expression one can use (2.2) to change the order in the fifth tensor factor as well as inside \mathcal{R} , to recover our calculation of $(\text{id} \otimes \Delta)\Delta(x \otimes k \otimes y)$ up to a change of basis labels. \square

Lemma 3.5. *The antipode of $B^{\text{op}} \bowtie A \bowtie B^*$ in Theorem 3.1 is given by*

$$\begin{aligned}
S(x \otimes k \otimes y) &= \bar{S}(e_{a(1)}^{(\infty)} \cdot_{\text{op}} x^{(\infty)} \cdot_{\text{op}} \bar{S}e_{a(3)}^{(\infty)}) \otimes S(x^{(\bar{1})} k_{(2)} f^{a(\bar{1})}) \otimes \underline{S}f^{a(\bar{0})} \\
&\quad \mathcal{R}(f^{a(\bar{1})}, S(x^{(\bar{1})} k_{(1)})) \mathcal{R}(S^2(k_{(5)} y^{(\bar{1})}), e_{a(1)}^{(\bar{1})} x^{(\bar{1})} e_{a(3)}^{(\bar{1})}) \\
&\quad \mathcal{R}(e_{a(1)}^{(\bar{1})}, x^{(\bar{1})} k_{(3)}) \mathcal{R}(S(k_{(4)} y^{(\bar{1})}), e_{a(3)}^{(\bar{1})}) \langle y^{(\bar{0})}, e_{a(2)} \rangle \\
&\quad v(f^{a(\bar{1})}) \bar{u}(e_{a(1)}^{(\bar{1})} x^{(\bar{1})} e_{a(3)}^{(\bar{1})}),
\end{aligned}$$

where $v(k) = \mathcal{R}(k_{(1)}, S k_{(2)})$ and $\bar{u}(k) = \mathcal{R}(S^2 k_{(1)}, k_{(2)})$.

Proof. We first compute $(S(x \otimes k \otimes y)_{(1)})(x \otimes k \otimes y)_{(2)}$, which on expanding out the product has in the first tensor factor

$$\begin{aligned}
&\bar{S}(e_{b(1)}^{(\infty)} \cdot_{\text{op}} x_{(1)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(3)}^{(\infty)}) \cdot_{\text{op}} e_{a(1)}^{(\infty)} \cdot_{\text{op}} x_{(2)}^{(\infty)} \cdot_{\text{op}} \bar{S}e_{a(3)}^{(\infty)} \\
&= (\underline{\epsilon}(e_{a(1)}) \underline{\epsilon}(e_{a(3)}) \underline{\epsilon}(x) \underline{\epsilon}(e_{b(1)}) \underline{\epsilon}(e_{b(3)}))^{(\infty)},
\end{aligned}$$

which further collapses the full expression to give

$$\begin{aligned}
&(S(x \otimes k \otimes y)_{(1)})(x \otimes k \otimes y)_{(2)} = \underline{\epsilon}(x) \otimes S(k_{(2)} f^{b(\bar{1})}) k_{(3)} y_{(1)}^{(\bar{1})} \otimes (\underline{S}f^{b(\bar{0})}) y_{(2)} \\
&\quad \langle f^a, e_b \rangle \langle y_{(1)}^{(\bar{0})}, e_a \rangle \mathcal{R}(f^{b(\bar{1})}, S k_{(1)}) \mathcal{R}(f^{b(\bar{1})}, k_{(4)} y_{(1)}^{(\bar{1})}) v(f^{b(\bar{1})}) \\
&= \underline{\epsilon}(x) \otimes (S y_{(1)}^{(\bar{0})}) (S k_{(2)} k_{(3)} y_{(1)}^{(\bar{1})}) \otimes (\underline{S} y_{(1)}^{(\bar{0})}) y_{(2)} \\
&\quad \mathcal{R}(y_{(1)}^{(\bar{0})}, S k_{(1)}) \mathcal{R}(y_{(1)}^{(\bar{0})}, y_{(1)}^{(\bar{1})}) \mathcal{R}(y_{(1)}^{(\bar{0})}, k_{(4)}) v(y_{(1)}^{(\bar{0})}) \\
&= \underline{\epsilon}(x) \otimes \underline{\epsilon}(k) (S y_{(1)}^{(\bar{1})}) y_{(1)}^{(\bar{1})} \otimes (\underline{S} y_{(1)}^{(\bar{0})}) y_{(2)} \mathcal{R}(y_{(1)}^{(\bar{1})}, y_{(1)}^{(\bar{1})}) v(y_{(1)}^{(\bar{1})}) \\
&= \underline{\epsilon}(x) \otimes \underline{\epsilon}(k) \otimes (\underline{S} y_{(1)}^{(\bar{0})}) y_{(2)} \mathcal{R}(y_{(1)}^{(\bar{1})}, y_{(1)}^{(\bar{1})}) v(y_{(1)}^{(\bar{1})}) \\
&= \underline{\epsilon}(x) \otimes \underline{\epsilon}(k) \otimes \underline{\epsilon}(y) = \underline{\epsilon}(x \otimes k \otimes y).
\end{aligned}$$

Similarly, on computing $(x \otimes k \otimes y)_{(1)}(S(x \otimes k \otimes y)_{(2)})$ we have $f^{a(\bar{0})} \underline{S}f^{b(\bar{0})} = (\underline{\epsilon} f^a \underline{\epsilon} f^b)^{(\bar{0})}$ in the third tensor factor which collapses the expressions to give

$$\begin{aligned}
&(x \otimes k \otimes y)_{(1)}(S(x \otimes k \otimes y)_{(2)}) = x_{(1)} \cdot_{\text{op}} \bar{S}x_{(2)}^{(\infty)} \otimes x_{(2)}^{(\bar{1})} k_{(2)} S(x_{(2)}^{(\bar{1})} k_{(3)}) \otimes \underline{\epsilon} y \\
&\quad \mathcal{R}(S^2 k_{(4)}, x_{(2)}^{(\bar{1})}) \mathcal{R}(S(x_{(2)}^{(\bar{1})} k_{(1)}), x_{(2)}^{(\infty)}) \bar{u}(x_{(2)}^{(\bar{1})}) \\
&= x_{(1)} \cdot_{\text{op}} \bar{S}x_{(2)}^{(\infty)} \otimes x_{(2)}^{(\bar{1})} k_{(2)} S k_{(3)} S x_{(2)}^{(\bar{1})} \otimes \underline{\epsilon} y \mathcal{R}(S^2 k_{(2)}, x_{(2)}^{(\bar{1})})
\end{aligned}$$

$$\begin{aligned}
& \mathcal{R}(Sk_{(1)}Sx_{(2)}^{\bar{(1)}}_{(1)}, x_{(2)}^{\bar{(1)}}_{(6)})\bar{u}(x_{(2)}^{\bar{(1)}}_{(4)}) \\
&= x_{(1)} \cdot_{\text{op}} \bar{S}x_{(2)}^{\bar{(\infty)}} \otimes 1 \otimes \epsilon y \mathcal{R}(S^2k_{(2)}, x_{(2)}^{\bar{(1)}}_{(3)}) \\
& \mathcal{R}(Sk_{(1)}, x_{(2)}^{\bar{(1)}}_{(4)})\mathcal{R}(Sx_{(2)}^{\bar{(1)}}_{(1)}, x_{(2)}^{\bar{(1)}}_{(5)})\bar{u}(x_{(2)}^{\bar{(1)}}_{(2)}) \\
&= x_{(1)} \cdot_{\text{op}} \bar{S}x_{(2)}^{\bar{(\infty)}} \otimes \epsilon k \otimes \epsilon y \mathcal{R}(Sx_{(2)}^{\bar{(1)}}_{(1)}, x_{(2)}^{\bar{(1)}}_{(3)})\bar{u}(x_{(2)}^{\bar{(1)}}_{(2)}) = \epsilon(x \otimes k \otimes y).
\end{aligned}$$

□

Finally, we show that the co-double bosonisation is coquasitriangular so as to have an inductive construction of such Hopf algebras.

Proposition 3.6. *The co-double bosonisation $B^{\text{op}} \bowtie A \bowtie B^*$ is coquasitriangular with*

$$\mathcal{R}(x \otimes k \otimes y, w \otimes \ell \otimes z) = \langle \underline{S}z^{\bar{(0)}}, x \rangle \mathcal{R}(k, \ell z^{\bar{(1)}}) \epsilon(y) \epsilon(w)$$

for all $x, w \in B^{\text{op}}$, $k, \ell \in A$, and $y, z \in B^*$.

Proof. (i) Expanding the definitions of the product and the coquasitriangular structure,

$$\begin{aligned}
& \mathcal{R}\left((m \otimes j \otimes v), (x \otimes k \otimes y)(w \otimes \ell \otimes z)\right) \\
&= \langle \underline{S}(y^{\bar{(0)}} z^{\bar{(0)}}), m \rangle \mathcal{R}(j, k \ell_{(1)} y^{\bar{(0)}} z^{\bar{(1)}}) \mathcal{R}(y^{\bar{(1)}}, \ell_{(2)}) \\
&= \langle \underline{S}(y^{\bar{(0)}} z^{\bar{(0)}}), m \rangle \mathcal{R}(j, k \ell_{(1)} y^{\bar{(1)}}_{(1)} z^{\bar{(0)}}) \mathcal{R}(y^{\bar{(1)}}_{(2)}, \ell_{(2)}) \\
&= \langle \underline{S}z^{\bar{(0)}} \bar{S}y^{\bar{(0)}}, m \rangle \mathcal{R}(j, k \ell_{(1)} y^{\bar{(1)}}_{(1)} z^{\bar{(0)}}) \mathcal{R}(y^{\bar{(1)}}_{(2)}, \ell_{(2)}) \mathcal{R}(y^{\bar{(0)}}_{(1)}, z^{\bar{(0)}}_{(1)}) \\
&= \langle \underline{S}z^{\bar{(0)}}, m_{(1)} \rangle \langle \underline{S}y^{\bar{(0)}}, m_{(2)} \rangle \mathcal{R}(j, k \ell_{(1)} y^{\bar{(1)}}_{(2)} z^{\bar{(1)}}_{(2)}) \mathcal{R}(y^{\bar{(1)}}_{(3)}, \ell_{(2)}) \mathcal{R}(y^{\bar{(1)}}_{(1)}, z^{\bar{(1)}}_{(1)}) \\
&= \langle \underline{S}z^{\bar{(0)}}, m_{(1)} \rangle \langle \underline{S}y^{\bar{(0)}}, m_{(2)} \rangle \mathcal{R}(j, k y^{\bar{(1)}}_{(3)} \ell_{(2)} z^{\bar{(1)}}_{(2)}) \mathcal{R}(y^{\bar{(1)}}, z^{\bar{(1)}}_{(1)}) \mathcal{R}(y^{\bar{(1)}}_{(2)}, \ell_{(1)}).
\end{aligned}$$

The second equality uses the right coaction on y . The third equality expands the braided-antipode $\underline{S}(y^{\bar{(0)}} z^{\bar{(0)}})$. The fourth equality uses the right-coaction on y and z , and evaluation. The last equality uses quasicommutativity to change the order of product inside the first \mathcal{R} . On the other side,

$$\begin{aligned}
& \mathcal{R}\left((m \otimes j \otimes v)_{(1)}, w \otimes \ell \otimes z\right) \mathcal{R}\left((m \otimes j \otimes v)_{(2)}, x \otimes k \otimes y\right) \\
&= \mathcal{R}(m_{(1)} \otimes m_{(2)}^{\bar{(1)}}_{(1)} j_{(1)} \otimes f^a, w \otimes \ell \otimes z) \\
& \mathcal{R}(e_{a(1)}^{\bar{(\infty)}} \cdot_{\text{op}} m_{(2)}^{\bar{(\infty)}} \cdot_{\text{op}} \bar{S}e_{a(3)}^{\bar{(\infty)}} \otimes j_{(4)} v_{(1)}^{\bar{(1)}}_{(2)} \otimes v_{(2)}, x \otimes k \otimes y) \\
& \mathcal{R}(e_{a(1)}^{\bar{(1)}}, m_{(2)}^{\bar{(1)}}_{(2)} j_{(2)}) \mathcal{R}(S(j_{(3)} v_{(1)}^{\bar{(1)}}_{(1)}), e_{a(3)}^{\bar{(1)}}) \langle v_{(1)}^{\bar{(0)}}, e_{a(2)} \rangle \\
&= \langle \underline{S}z^{\bar{(0)}}, m_{(1)} \rangle \langle \underline{S}y^{\bar{(0)}}, m_{(2)}^{\bar{(\infty)}} \rangle \mathcal{R}(m_{(2)}^{\bar{(1)}}_{(1)} j_{(1)}, \ell z^{\bar{(1)}}) \mathcal{R}(j_{(2)}, k y^{\bar{(1)}}) \\
&= \langle \underline{S}z^{\bar{(0)}}, m_{(1)} \rangle \langle \underline{S}y^{\bar{(0)}}, m_{(2)} \rangle \mathcal{R}(y^{\bar{(0)}}_{(1)} j_{(1)}, \ell z^{\bar{(1)}}) \mathcal{R}(j_{(2)}, k y^{\bar{(1)}}) \\
&= \langle \underline{S}z^{\bar{(0)}}, m_{(1)} \rangle \langle \underline{S}y^{\bar{(0)}}, m_{(2)} \rangle \mathcal{R}(y^{\bar{(1)}}_{(1)} j_{(1)}, \ell z^{\bar{(1)}}) \mathcal{R}(j_{(2)}, k y^{\bar{(1)}}_{(2)}).
\end{aligned}$$

The third equality uses $\langle y^{\bar{(0)}}, m_{(2)}^{\bar{(\infty)}} \rangle m_{(2)}^{\bar{(1)}} = \langle y^{\bar{(0)}}, m_{(2)} \rangle y^{\bar{(0)}}_{(1)}$ and the fourth uses the right coaction on y . We can then use (2.2) to gather the parts of j and obtain the same expression as on the first side. (ii) Similarly expanding the definitions,

$$\begin{aligned}
& \mathcal{R}\left((x \otimes k \otimes y)(w \otimes \ell \otimes z), (m \otimes j \otimes v)\right) \\
&= \langle \underline{S}v^{\bar{(0)}}, x \cdot_{\text{op}} w^{\bar{(\infty)}} \rangle \mathcal{R}(k_{(2)} \ell, j v^{\bar{(1)}}) \mathcal{R}(Sk_{(1)}, w^{\bar{(1)}}) \\
&= \langle \underline{S}v^{\bar{(0)}}, w^{\bar{(\infty)}} \bar{S}x^{\bar{(\infty)}} \rangle \mathcal{R}(k_{(2)} \ell, j v^{\bar{(1)}}) \mathcal{R}(Sk_{(1)}, w^{\bar{(1)}}) \mathcal{R}(Sx^{\bar{(1)}}, w^{\bar{(\infty)}}_{(1)}) \\
&= \langle \underline{S}v^{\bar{(0)}}_{(1)} \bar{S}x^{\bar{(\infty)}}, w^{\bar{(\infty)}} \rangle \langle \underline{S}v^{\bar{(0)}}_{(2)}, w^{\bar{(\infty)}} \rangle \mathcal{R}(k_{(2)} \ell, j v^{\bar{(1)}}) \mathcal{R}(Sk_{(1)}, w^{\bar{(1)}}_{(1)}) \\
& \mathcal{R}(Sx^{\bar{(1)}}, w^{\bar{(1)}}_{(2)}) \mathcal{R}(v^{\bar{(0)}}_{(1)} \bar{S}x^{\bar{(\infty)}}, v^{\bar{(0)}}_{(2)} \bar{S}x^{\bar{(\infty)}}) \\
&= \langle \underline{S}v^{\bar{(0)}}_{(1)} \bar{S}x^{\bar{(\infty)}}, w^{\bar{(\infty)}} \rangle \langle \underline{S}v^{\bar{(0)}}_{(2)}, w^{\bar{(\infty)}} \rangle \mathcal{R}(k_{(2)} \ell, j v_{(1)}^{\bar{(1)}}_{(2)} v_{(2)}^{\bar{(1)}}_{(2)}) \mathcal{R}(Sk_{(1)}, w^{\bar{(1)}}_{(1)}) \\
& \mathcal{R}(Sx^{\bar{(1)}}, w^{\bar{(1)}}_{(2)}) \mathcal{R}(v_{(1)}^{\bar{(1)}}_{(1)}, v_{(2)}^{\bar{(1)}}_{(1)}) \\
&= \langle \underline{S}v^{\bar{(0)}}_{(1)} \bar{S}x^{\bar{(\infty)}}, w^{\bar{(\infty)}} \rangle \langle \underline{S}v^{\bar{(0)}}_{(2)}, w^{\bar{(\infty)}} \rangle \mathcal{R}(k_{(2)} \ell, j v_{(1)}^{\bar{(1)}}_{(2)} v_{(2)}^{\bar{(1)}}_{(2)}) \mathcal{R}(Sk_{(1)}, v_{(2)}^{\bar{(0)}}_{(1)}) \\
& \mathcal{R}(Sv_{(1)}^{\bar{(0)}}_{(1)}, v_{(2)}^{\bar{(0)}}_{(2)}) \mathcal{R}(v_{(1)}^{\bar{(1)}}_{(1)}, v_{(2)}^{\bar{(1)}}_{(1)})
\end{aligned}$$

$$\begin{aligned}
&= \langle \underline{S}v_{(1)}^{(\bar{0})}, x \rangle \langle \underline{S}v_{(2)}^{(\bar{0})}, w \rangle \mathcal{R}(k_{(2)}\ell, jv_{(1)}^{(\bar{1})}v_{(2)}^{(\bar{1})}) \mathcal{R}(Sk_{(1)}, v_{(2)}^{(\bar{1})}) \\
&\quad \mathcal{R}(Sv_{(1)}^{(\bar{1})}, v_{(2)}^{(\bar{1})}) \mathcal{R}(v_{(1)}^{(\bar{1})}, v_{(2)}^{(\bar{1})}) \\
&= \langle \underline{S}v_{(1)}^{(\bar{0})}, x \rangle \langle \underline{S}v_{(2)}^{(\bar{0})}, w \rangle \mathcal{R}(k, j_{(1)}v_{(1)}^{(\bar{1})}) \mathcal{R}(\ell, j_{(2)}v_{(1)}^{(\bar{1})}v_{(2)}^{(\bar{1})}).
\end{aligned}$$

The second equality expands the braided-product \cdot_{op} . The third equality uses the left-coaction on w , followed by the duality pairing and taking \underline{S} to the left in $\underline{\Delta}(Sv_{(1)}^{(\bar{0})})$. The fourth equality uses the comodule coalgebra property (2.4) on v and the right coaction axioms. The fifth equality moves the coactions onto x, w by duality. The sixth equality is similar to the fourth. For the last equality we cancel the last two \mathcal{R} s and use (2.1) to gather k inside \mathcal{R} and cancel further. On the other side,

$$\begin{aligned}
&\mathcal{R}\left((x \otimes k \otimes y), (m \otimes j \otimes v)_{(1)}\right) \mathcal{R}\left((w \otimes \ell \otimes z), (m \otimes j \otimes v)_{(2)}\right) \\
&= \langle Sf^{a(\bar{0})}, x \rangle \langle \underline{S}v_{(2)}^{(\bar{0})}, w \rangle \langle v_{(1)}^{(\bar{0})}, e_a \rangle \mathcal{R}(k, j_{(1)}f^{a(\bar{1})}) \mathcal{R}(\ell, j_{(2)}v_{(1)}^{(\bar{1})}v_{(2)}^{(\bar{1})}) \\
&= \langle \underline{S}v_{(1)}^{(\bar{0})}, x \rangle \langle \underline{S}v_{(2)}^{(\bar{0})}, w \rangle \mathcal{R}(k, j_{(1)}v_{(1)}^{(\bar{0})}) \mathcal{R}(\ell, j_{(2)}v_{(1)}^{(\bar{1})}v_{(2)}^{(\bar{1})})
\end{aligned}$$

on substituting $f^a = v_{(1)}^{(\bar{0})}$. We can then use the right coaction property on $v_{(1)}$ to recover the result of our first calculation. (iii) We expand the definitions to compute

$$\begin{aligned}
&(x \otimes k \otimes y)_{(2)}(w \otimes \ell \otimes z)_{(2)} \mathcal{R}\left((x \otimes k \otimes y)_{(1)}, (w \otimes \ell \otimes z)_{(2)}\right) \\
&= x_{(2)}^{(\infty)} \cdot_{\text{op}} e_{b(1)}^{(\infty)} \cdot_{\text{op}} w^{(\infty)} \cdot_{\text{op}} \bar{S}e_{b(3)}^{(\infty)} \otimes k_{(3)}\ell_{(4)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \\
&\quad \mathcal{R}(Sk_{(2)}, e_{b(1)}^{(\bar{1})}w^{(\bar{1})}e_{b(3)}^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(5)}z_{(1)}^{(\bar{1})}) \mathcal{R}(e_{b(1)}^{(\bar{1})}, w^{(\bar{1})}\ell_{(2)}) \\
&\quad \mathcal{R}(S(\ell_{(3)}z_{(1)}^{(\bar{1})}), e_{b(3)}^{(\bar{1})}) \mathcal{R}(x_{(2)}^{(\bar{1})}k_{(1)}, w^{(\bar{1})}\ell_{(1)}f^{b(\bar{1})}) \langle Sf^{b(\bar{0})}, x_{(1)} \rangle \langle z_{(1)}^{(\bar{0})}, e_{b(2)} \rangle \\
&= x_{(4)}^{(\infty)} \cdot_{\text{op}} \bar{S}^{-1}x_{(3)}^{(\infty)} \cdot_{\text{op}} w^{(\infty)} \cdot_{\text{op}} \bar{S}\bar{S}^{-1}x_{(1)}^{(\infty)} \otimes k_{(3)}\ell_{(4)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \\
&\quad \mathcal{R}(Sk_{(2)}, x_{(3)}^{(\bar{1})}w^{(\bar{1})}x_{(1)}^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(5)}z_{(1)}^{(\bar{1})}) \mathcal{R}(x_{(3)}^{(\bar{1})}, w^{(\bar{1})}\ell_{(2)}) \\
&\quad \mathcal{R}(S(\ell_{(3)}z_{(1)}^{(\bar{1})}), x_{(1)}^{(\bar{1})}) \mathcal{R}(x_{(4)}^{(\bar{1})}k_{(1)}, w^{(\bar{1})}\ell_{(1)}x_{(1)}^{(\bar{1})}x_{(2)}^{(\bar{1})}x_{(3)}^{(\bar{1})}) \\
&\quad \mathcal{R}(x_{(2)}^{(\bar{1})}x_{(3)}^{(\bar{1})}, x_{(1)}^{(\bar{1})}) \mathcal{R}(x_{(3)}^{(\bar{1})}, x_{(2)}^{(\bar{1})}) \langle z_{(1)}^{(\bar{0})}, \bar{S}^{-1}x_{(2)}^{(\infty)} \rangle \\
&= w^{(\infty)} \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes k_{(3)}\ell_{(3)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \\
&\quad \mathcal{R}(Sk_{(2)}, w^{(\bar{1})}x_{(1)}^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(4)}z_{(1)}^{(\bar{1})}) \mathcal{R}(S(\ell_{(2)}z_{(1)}^{(\bar{1})}), x_{(1)}^{(\bar{1})}) \\
&\quad \mathcal{R}(k_{(1)}, w^{(\bar{1})}\ell_{(1)}x_{(1)}^{(\bar{1})}x_{(2)}^{(\bar{1})}) \mathcal{R}(x_{(2)}^{(\bar{1})}, x_{(1)}^{(\bar{1})}) \langle z_{(1)}^{(\bar{0})}, \bar{S}^{-1}x_{(2)}^{(\infty)} \rangle \\
&= w^{(\infty)} \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes k_{(3)}\ell_{(3)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \\
&\quad \mathcal{R}(Sk_{(2)}, w^{(\bar{1})}x_{(1)}^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(4)}z_{(1)}^{(\bar{1})}) \mathcal{R}(S(\ell_{(2)}z_{(1)}^{(\bar{1})}), x_{(1)}^{(\bar{1})}) \\
&\quad \mathcal{R}(k_{(1)}, w^{(\bar{1})}\ell_{(1)}x_{(1)}^{(\bar{1})}z_{(1)}^{(\bar{0})}) \mathcal{R}(z_{(1)}^{(\bar{0})}, x_{(1)}^{(\bar{1})}) \langle \underline{S}z_{(1)}^{(\bar{0})}, x_{(2)} \rangle \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes k_{(2)}\ell_{(3)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \langle \underline{S}z_{(1)}^{(\bar{0})}, x_{(2)} \rangle \\
&\quad \mathcal{R}(Sk_{(2)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(y^{(\bar{1})}, \ell_{(4)}z_{(1)}^{(\bar{1})}) \mathcal{R}(S\ell_{(2)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(k_{(1)}, \ell_{(1)}x_{(1)}^{(\bar{1})}z_{(1)}^{(\bar{1})}) \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes k_{(2)}\ell_{(3)}z_{(1)}^{(\bar{1})} \otimes y^{(\bar{0})}z_{(2)} \langle \underline{S}z_{(1)}^{(\bar{0})}, x_{(2)} \rangle \\
&\quad \mathcal{R}(y^{(\bar{1})}, \ell_{(4)}z_{(1)}^{(\bar{1})}) \mathcal{R}(S\ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(k_{(1)}, \ell_{(2)}z_{(1)}^{(\bar{1})}) \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)}z_{(1)}^{(\bar{1})}k_{(1)} \otimes y^{(\bar{0})}z_{(2)} \langle \underline{S}z_{(1)}^{(\bar{0})}, x_{(2)} \rangle \\
&\quad \mathcal{R}(y^{(\bar{1})}, \ell_{(4)}z_{(1)}^{(\bar{1})}) \mathcal{R}(S\ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(k_{(2)}, \ell_{(3)}z_{(1)}^{(\bar{1})}).
\end{aligned}$$

The second equality uses the duality $\langle Sf^{b(\bar{0})}, x_{(1)} \rangle f^{b(\bar{1})} = \langle f^b, \bar{S}^{-1}x_{(1)}^{(\infty)} \rangle x_{(1)}^{(\bar{1})}$ to substitute $e_b = \bar{S}^{-1}x_{(1)}^{(\infty)}$ in all the places where it occurs, and the comodule algebra property (2.3). The third equality cancels $(x_{(4)} \cdot_{\text{op}} \bar{S}^{-1}x_{(3)}^{(\infty)})^{(\infty)}$ resulting in trivial coactions. We use the duality $\langle z_{(1)}^{(\bar{0})}, \bar{S}^{-1}x_{(2)}^{(\infty)} \rangle x_{(2)}^{(\bar{1})} = \langle \underline{S}z_{(1)}^{(\bar{0})}, x_{(2)} \rangle z_{(1)}^{(\bar{0})}$ for the fourth equality and gather $w^{(\bar{1})}$ inside \mathcal{R} to cancel it for the fifth one. The sixth equality uses (2.2) in

$$\ell_{(1)}x_{(1)}^{(\bar{1})}\bar{\mathcal{R}}(x_{(1)}^{(\bar{1})}, \ell_{(2)}) = x_{(1)}^{(\bar{1})}\ell_{(2)}\bar{\mathcal{R}}(x_{(1)}^{(\bar{1})}, \ell_{(2)})$$

and then gathers the parts of $x_{(1)}^{(\bar{1})}$, and cancels some \mathcal{R} s. We finally use (2.2) to change the order of products in the third tensor factor. On the other side,

$$\begin{aligned}
& (w \otimes \ell \otimes z)_{(1)} (x \otimes k \otimes y)_{(1)} \mathcal{R} \left((x \otimes k \otimes y)_{(2)}, (w \otimes \ell \otimes z)_{(2)} \right) \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)} x_{(2)}^{(\bar{1})} k_{(1)} \otimes f^{b(\bar{0})} f^a \langle \underline{S} z_{(2)}^{(\bar{0})}, e_{a(1)}^{(\infty)} \rangle \cdot_{\text{op}} x_{(2)}^{(\infty)} \cdot_{\text{op}} \bar{S} e_{a(3)}^{(\infty)} \rangle \\
& \quad \mathcal{R}(k_{(5)} y_{(2)}^{(\bar{1})}, \ell_{(3)} z_{(1)}^{(\bar{1})} z_{(2)}^{(\bar{1})}) \mathcal{R}(S \ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(f^{b(\bar{1})}, x_{(2)}^{(\bar{1})} k_{(2)}) \mathcal{R}(e_{a(1)}^{(\bar{1})}, x_{(2)}^{(\bar{1})} k_{(3)}) \\
& \quad \mathcal{R}(S(k_{(4)} y_{(1)}^{(\bar{1})}), e_{a(3)}^{(\bar{1})}) \langle y^{(\bar{0})}, e_{a(2)} \rangle \langle z_{(1)}^{(\bar{0})}, e_b \rangle \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)} x_{(2)}^{(\bar{1})} k_{(1)} \otimes z_{(1)}^{(\bar{0})} f^a \langle \underline{S} z_{(2)}^{(\bar{0})}, (\underline{S}^{-1} e_{a(3)}^{(\infty)}) x_{(2)}^{(\infty)} e_{a(1)}^{(\infty)} \rangle \\
& \quad \mathcal{R}(k_{(5)} y_{(2)}^{(\bar{1})}, \ell_{(3)} z_{(1)}^{(\bar{1})} z_{(2)}^{(\bar{1})}) \mathcal{R}(S \ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(z_{(1)}^{(\bar{1})} k_{(1)}, x_{(2)}^{(\bar{1})} k_{(2)}) \\
& \quad \mathcal{R}(e_{a(1)}^{(\bar{1})} k_{(3)}, x_{(2)}^{(\bar{1})} k_{(3)}) \mathcal{R}(S(k_{(4)} y_{(1)}^{(\bar{1})}), e_{a(3)}^{(\bar{1})}) \mathcal{R}(S e_{a(1)}^{(\bar{1})} k_{(2)}, x_{(2)}^{(\bar{1})} k_{(4)}) \\
& \quad \mathcal{R}(S(x_{(2)}^{(\bar{1})} e_{a(1)}^{(\bar{1})} e_{a(3)}^{(\bar{1})}) \langle y^{(\bar{0})}, e_{a(2)} \rangle \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)} x_{(2)}^{(\bar{1})} k_{(1)} \otimes z_{(1)}^{(\bar{0})} f^a \mathcal{R}(S \ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(z_{(1)}^{(\bar{1})} k_{(1)}, x_{(2)}^{(\bar{1})} k_{(2)}) \\
& \quad \mathcal{R}(z_{(2)}^{(\bar{1})} k_{(3)}, x_{(2)}^{(\bar{1})} k_{(3)}) \mathcal{R}(S(k_{(4)} y_{(1)}^{(\bar{1})}), z_{(1)}^{(\bar{1})} k_{(1)}) \mathcal{R}(z_{(1)(2)}^{(\bar{1})} k_{(2)}, x_{(2)}^{(\bar{1})} k_{(4)}) \\
& \quad \mathcal{R}(S(x_{(2)}^{(\bar{1})} e_{a(1)}^{(\bar{1})} e_{a(3)}^{(\bar{1})}), z_{(4)}^{(\bar{1})} k_{(2)}) \mathcal{R}(k_{(5)} y_{(2)}^{(\bar{1})}, \ell_{(3)} z_{(1)}^{(\bar{1})} z_{(2)}^{(\bar{1})} e_{a(3)}^{(\bar{1})} z_{(3)}^{(\bar{1})} z_{(4)}^{(\bar{1})}) \\
& \quad \mathcal{R}(z_{(3)}^{(\bar{1})} k_{(1)}, z_{(4)}^{(\bar{1})} k_{(3)}) \mathcal{R}(z_{(2)}^{(\bar{1})} k_{(4)} z_{(4)}^{(\bar{1})}) \langle \underline{S} z_{(2)}^{(\bar{0})}, e_{a(1)}^{(\infty)} \rangle \langle y^{(\bar{0})}, e_{a(2)} \rangle \langle \underline{S} z_{(4)}^{(\bar{0})}, \underline{S}^{-1} e_{a(3)}^{(\infty)} \rangle \\
& \quad \langle \underline{S} z_{(3)}^{(\bar{0})}, x_{(2)}^{(\infty)} \rangle \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)} x_{(2)}^{(\bar{1})} k_{(1)} \otimes y^{(\bar{0})} z_{(2)}^{(\bar{0})} \mathcal{R}(S \ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(S(k_{(2)} y_{(1)}^{(\bar{1})}), z_{(2)}^{(\bar{1})} k_{(1)}) \\
& \quad \mathcal{R}(S(x_{(2)}^{(\bar{1})} k_{(2)}, z_{(2)}^{(\bar{1})} k_{(2)}) \mathcal{R}(k_{(3)} y_{(2)}^{(\bar{1})} z_{(1)}^{(\bar{1})} z_{(2)}^{(\bar{1})} k_{(4)}) \mathcal{R}(z_{(1)}^{(\bar{1})} k_{(1)}, z_{(2)}^{(\bar{1})} k_{(3)}) \langle \underline{S} z_{(1)}^{(\bar{0})}, x_{(2)}^{(\infty)} \rangle \\
&= w \cdot_{\text{op}} x_{(1)}^{(\infty)} \otimes \ell_{(2)} z_{(1)}^{(\bar{1})} k_{(1)} \otimes y^{(\bar{0})} z_{(2)}^{(\bar{0})} \mathcal{R}(S \ell_{(1)}, x_{(1)}^{(\bar{1})}) \mathcal{R}(S(k_{(2)} y_{(1)}^{(\bar{1})}), z_{(2)}^{(\bar{1})} k_{(1)}) \\
& \quad \mathcal{R}(k_{(3)} y_{(2)}^{(\bar{1})}, \ell_{(3)} z_{(1)}^{(\bar{1})} z_{(2)}^{(\bar{1})} k_{(4)}) \mathcal{R}(S z_{(1)}^{(\bar{1})} k_{(2)}, z_{(2)}^{(\bar{1})} k_{(2)}) \mathcal{R}(z_{(1)}^{(\bar{1})} k_{(3)}, z_{(2)}^{(\bar{1})} k_{(3)}) \langle \underline{S} z_{(1)}^{(\bar{0})}, x_{(1)}^{(\infty)} \rangle.
\end{aligned}$$

The second equality uses $\langle z_{(1)}^{(\bar{0})}, e_b \rangle$ to substitute $f^b = z_{(1)}^{(\bar{0})}$ and we then expand \cdot_{op} inside the pairing. For the third equality, we use

$$\begin{aligned}
& \langle \underline{S} z_{(2)}^{(\bar{0})}, (\underline{S}^{-1} e_{a(3)}^{(\infty)}) x_{(2)}^{(\infty)} e_{a(1)}^{(\infty)} \rangle \\
&= \langle (\underline{S} z_{(2)}^{(\bar{0})})_{(1)}, \underline{S}^{-1} e_{a(3)}^{(\infty)} \rangle \langle (\underline{S} z_{(2)}^{(\bar{0})})_{(2)}, x_{(2)}^{(\infty)} \rangle \langle (\underline{S} z_{(2)}^{(\bar{0})})_{(3)}, e_{a(1)}^{(\infty)} \rangle
\end{aligned}$$

and move \underline{S} to the left in $\underline{\Delta}^2(\underline{S} z_{(2)}^{(\bar{0})})$. For the fourth equality we gather the coproducts of e_a to give $\langle (\underline{S} z_{(2)}^{(\bar{0})}) y^{(\bar{0})} z_{(4)}^{(\bar{0})}, e_a \rangle$ so that we can set $f^a = (\underline{S} z_{(2)}^{(\bar{0})}) y^{(\bar{0})} z_{(4)}^{(\bar{0})}$, allowing us to cancel $(z_{(1)} \underline{S} z_{(2)})^{(\bar{0})}$ and drop out following coactions. For the fifth equality, we use the duality pairing $\langle \underline{S} z_{(1)}^{(\bar{0})}, x_{(2)}^{(\infty)} \rangle x_{(2)}^{(\bar{1})} = \langle \underline{S} z_{(1)}^{(\bar{0})}, x_{(2)}^{(\infty)} \rangle z_{(1)}^{(\bar{0})} x_{(2)}^{(\bar{1})}$ and then gather $z_{(2)}^{(\bar{1})}$ inside \mathcal{R} so as to cancel it and recover the result of our first calculation. \square

4. CONSTRUCTION OF $\mathfrak{c}_q[SL_2]$ BY CO-DOUBLE BOSONISATION

The coquasitriangular Hopf algebra $\mathbb{C}_q[SL_2]$ in some standard conventions is generated by a, b, c, d with the relations,

$$\begin{aligned}
ba &= qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \\
ad - q^{-1}bc &= 1, \quad da - ad = (q - q^{-1})bc,
\end{aligned}$$

a ‘matrix’ form of coproduct (so $\Delta a = a \otimes a + b \otimes c$ etc.), $\epsilon(a) = \epsilon(d) = 1$, $\epsilon(b) = \epsilon(c) = 0$ and antipode $Sa = d$, $Sd = a$, $Sb = -qb$, $Sc = -q^{-1}$. The reduced version $c_q[SL_2]$ has

$$a^n = 1 = d^n, \quad b^n = 0 = c^n$$

as additional relations when q is a primitive n -th root of unity. We will show how some version of this is obtained by co-double bosonisation. Let $A = \mathbb{C}_q[t]/(t^n - 1)$ be a coquasitriangular Hopf algebra with t grouplike and $\mathcal{R}(t^r, t^s) = q^{rs}$. Also let $B = \mathbb{C}[X]/(X^n)$ be a braided group in ${}^A\mathcal{M}$ with

$$\Delta_L X = t \otimes X, \quad \underline{\Delta} X = 1 \otimes X + X \otimes 1, \quad \underline{\epsilon} X = 0, \quad \underline{S} X = -X, \quad \Psi(X^r \otimes X^s) = q^{rs} X^s \otimes X^r.$$

The dual $B^* = \mathbb{C}[Y]/(Y^n)$ lives in \mathcal{M}^A with the same form of coproduct, etc., as for B , but with right-action $\Delta_R Y = Y \otimes t$. We choose pairing $\langle X, Y \rangle = 1$ and take a basis of B and a dual basis of B^* respectively as

$$\{e_a\} = \{X^a\}_{0 \leq a < n}, \quad \{f^a\} = \left\{ \frac{Y^a}{[a]_q!} \right\}_{0 \leq a < n},$$

where $[a]_q$ is a q -integers defined by $[a]_q = (1 - q^a)/(1 - q)$ and $[a]_q! = [a]_q [a-1]_q \cdots [1]_q!$ with $[0]_q! = 1$. We also write $\begin{bmatrix} a \\ r \end{bmatrix}_q = \frac{[a]_q!}{[r]_q! [a-r]_q!}$. We write $X^{a(\text{op})} = X \cdot_{\text{op}} X \cdot_{\text{op}} \cdots \cdot_{\text{op}} X$ with a -many X , and find inductively that

$$(4.1) \quad X^a = q^{\frac{a(a-1)}{2}} X^{a(\text{op})}, \quad \bar{S}(X^a) = (-1)^a X^{a(\text{op})}.$$

Theorem 4.1. *Let q be a primitive n -th root of unity and A, B, B^* be as above.*

- (1) *The co-double bosonisation of B , denoted $\mathfrak{c}_q[SL_2]$, has generators X, t, Y and $X^n = Y^n = 0, \quad t^n = 1, \quad YX = XY, \quad Xt = qtX, \quad Yt = qtY,$*

$$\begin{aligned} \Delta t &= q \sum_{a=0}^{n-2} (q-1)^{a-1} (1 - q^{-a-1}) t Y^a \otimes X^a t \\ &= \frac{t}{q-1} \left(q \frac{1 - ((q-1)Y \otimes X)^{n-1}}{1 - (q-1)Y \otimes X} - \frac{1 - ((1-q^{-1})Y \otimes X)^{n-1}}{1 - (1-q^{-1})Y \otimes X} \right) t, \\ \Delta X &= X \otimes 1 + \sum_{a=0}^{n-2} (q-1)^a t Y^a \otimes X^{a+1} = X \otimes 1 + t \left(\frac{1 - ((q-1)Y \otimes X)^{n-1}}{1 - (q-1)Y \otimes X} \right) X, \\ \Delta Y &= 1 \otimes Y + \sum_{a=0}^{n-2} (1 - q^{-1})^a Y^{a+1} \otimes X^a t = 1 \otimes Y + Y \left(\frac{1 - ((1-q^{-1})Y \otimes X)^{n-1}}{1 - (1-q^{-1})Y \otimes X} \right) t. \end{aligned}$$

- (2) *If $n = 2m + 1$, there is an isomorphism $\phi : \mathfrak{c}_q[SL_2] \rightarrow c_{q^{-m}}[SL_2]$ defined by*

$$\phi(X) = bd^{-1}, \quad \phi(t) = d^{-2}, \quad \phi(Y) = \frac{d^{-1}c}{q^m - q^{-m}}.$$

Proof. (1) First we determine the products

$$\begin{aligned} (1 \otimes 1 \otimes Y)(X \otimes 1 \otimes 1) &= X^{(\infty)} \otimes 1 \otimes Y^{(\bar{0})} \mathcal{R}(Y^{(\bar{1})}, 1) \mathcal{R}(S1, X^{(\bar{1})}) = X \otimes 1 \otimes Y, \\ (1 \otimes t \otimes 1)(X \otimes 1 \otimes 1) &= X^{(\infty)} \otimes t \otimes 1 \mathcal{R}(St, X^{(\bar{1})}) = q^{-1} X \otimes t \otimes 1, \\ (1 \otimes 1 \otimes Y)(1 \otimes t \otimes 1) &= 1 \otimes t \otimes Y^{(\bar{0})} \mathcal{R}(Y^{(\bar{1})}, t) = q1 \otimes t \otimes Y \end{aligned}$$

as stated. The algebra generated by X, Y, t with these relations is n^3 dimensional, hence these are all the relations we need. Before go further, we note the q -identities

$$(4.2) \quad \sum_{r=0}^a (-1)^r \frac{q^{\frac{r(r+1)}{2}}}{[r]_q! [a-r]_q!} = (1-q)^a, \quad \sum_{r=0}^a q^r (-1)^r \frac{q^{\frac{r(r+1)}{2}}}{[r]_q! [a-r]_q!} = (1-q)^a [a+1]_q.$$

Then, using (4.1), we compute

$$\begin{aligned} \Delta(1 \otimes t \otimes 1) &= \sum_{a=0}^{n-1} 1 \otimes t \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \begin{bmatrix} a \\ r \end{bmatrix}_q X^r \cdot_{\text{op}} \bar{S} X^{a-r} \otimes t \otimes 1 \mathcal{R}(t^r, t) \mathcal{R}(St, t^{a-r}) \right) \\ &= \sum_{a=0}^{n-1} 1 \otimes t \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \begin{bmatrix} a \\ r \end{bmatrix}_q (-1)^{a-r} q^{\frac{r(r-1)}{2} + 2r-a} X^{r(\text{op})} \cdot_{\text{op}} X^{(a-r)(\text{op})} \otimes t \otimes 1 \right) \\ &= \sum_{a=0}^{n-2} \sum_{r=0}^a \frac{(-1)^{a-r} \begin{bmatrix} a \\ r \end{bmatrix}_q q^{\frac{r(r+1)}{2}} q^{-a+r}}{[a]_q!} t Y^a \otimes X^a t \end{aligned}$$

since there is no contribution when $a = n - 1$. We then use (4.2). Similarly,

$$\begin{aligned} \Delta(X \otimes 1 \otimes 1) &= X \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\ &\quad + \sum_{a=0}^{n-1} 1 \otimes t \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \begin{bmatrix} a \\ r \end{bmatrix}_q X^r \cdot_{\text{op}} X \cdot_{\text{op}} \bar{S} X^{a-r} \otimes 1 \otimes 1 \mathcal{R}(t^r, t) \right) \end{aligned}$$

$$\begin{aligned}
&= X \otimes 1 \otimes 1 \otimes 1 \otimes 1 \\
&+ \sum_{a=0}^{n-1} 1 \otimes t \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \begin{bmatrix} a \\ r \end{bmatrix}_q q^{\frac{r(r+1)}{2}} X^{r(\text{op})} \cdot_{\text{op}} X \cdot_{\text{op}} (-1)^{a-r} X^{(a-r)(\text{op})} \otimes 1 \otimes 1 \right) \\
&= X \otimes 1 + \sum_{a=0}^{n-2} \sum_{r=0}^a \frac{(-1)^{a-r} \begin{bmatrix} a \\ r \end{bmatrix}_q q^{\frac{r(r+1)}{2}}}{[a]_q!} t Y^a \otimes X^{a+1},
\end{aligned}$$

where for $a = n - 1$, we will have the term $tY^{n-1} \otimes X^n = 0$. We again use (4.2).

Finally, we use $\underline{\Delta}^2(e_a) = \underline{\Delta}^2(X^a) = \sum_{r=0}^a \sum_{s=0}^r \begin{bmatrix} a \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q X^s \otimes X^{r-s} \otimes X^{a-r}$ to find

$$\begin{aligned}
\Delta(1 \otimes 1 \otimes Y) &= 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes Y \\
&+ \sum_{a=0}^{n-1} 1 \otimes 1 \otimes f^a \otimes e_{a(1)} \cdot_{\text{op}} \bar{S} e_{a(3)}^{(\infty)} \otimes t \otimes 1 \mathcal{R}(St, e_{a(3)}^{(\bar{1})}) \langle Y, e_{a(2)} \rangle. \\
&= 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes Y \\
&+ \sum_{a=0}^{n-1} 1 \otimes 1 \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \sum_{s=0}^r \begin{bmatrix} a \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q X^s \cdot_{\text{op}} \bar{S} X^{a-r} \otimes t \otimes 1 \mathcal{R}(St, t^{a-r}) \langle Y, X^{r-s} \rangle \right) \\
&= 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes Y \\
&+ \sum_{a=0}^{n-1} 1 \otimes 1 \otimes \frac{Y^a}{[a]_q!} \otimes \left(\sum_{r=0}^a \sum_{s=0}^r \begin{bmatrix} a \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q \delta_{1,r-s} q^{\frac{s(s-1)}{2}} q^{-a+r} X^{s(\text{op})} \cdot_{\text{op}} (-1)^{a-r} X^{(a-r)(\text{op})} \otimes t \otimes 1 \right) \\
&= 1 \otimes Y + \sum_{a=0}^{n-1} \sum_{r=0}^a \sum_{s=0}^r \frac{(-1)^{a-r} \delta_{1,r-s} \begin{bmatrix} a \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q q^{\frac{s(s-1)}{2}} q^{-a+r}}{[a]_q!} Y^a \otimes X^{a-r+s} t \\
&= 1 \otimes Y + \sum_{a=0}^{n-2} \sum_{r=0}^a \frac{(-1)^{a-r} \begin{bmatrix} a \\ r \end{bmatrix}_q q^{\frac{r(r+1)}{2}} q^{-a}}{[a]_q!} Y^{a+1} \otimes X^a t.
\end{aligned}$$

There was no contribution from $a = 0$ and for $a > 0$ we needed $s = r - 1$ for a contribution. We then use (4.2). The general theory in Section 3 ensures that the Hopf algebra is coquasitriangular.

(2) If $n = 2m + 1$ then $\varphi : c_{q^{-m}}[SL_2] \rightarrow \mathfrak{c}_q[SL_2]$ defined by

$$\varphi(a) = t^{m+1} + (q^m - q^{-m})Xt^mY, \quad \varphi(b) = Xt^m, \quad \varphi(c) = (q^m - q^{-m})t^mY, \quad \varphi(d) = t^m.$$

is an algebra map and inverse to ϕ . Tedious but straightforward calculation gives

$$\Delta(\varphi(d)) = \Delta t^m = t^m \otimes t^m + (q^{2m} - 1)t^mY \otimes t^mX = t^m \otimes t^m + (q^m - q^{-m})t^mY \otimes Xt^m,$$

to prove that $\Delta(\varphi(d)) = (\varphi \otimes \varphi)\Delta d$. The coalgebra map property on the other generators then follows using this formula for Δt^m . Furthermore, the coquasitriangular structure from Lemma 3.6 computed on $\varphi(a), \varphi(b), \varphi(c), \varphi(d)$ as a matrix $\varphi(t^i_j)$ is

$$(4.3) \quad R^I_J = q^{m(m+1)} \begin{pmatrix} q^{-m} & 0 & 0 & 0 \\ 0 & 1 & q^{-m} - q^m & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-m} \end{pmatrix}.$$

for the values of $\mathcal{R}(\varphi(t^i_j), \varphi(t^k_l))$ where $I = (i, k)$ is $(1,1), (1,2), (2,1)$, or $(2,2)$ and similarly for $J = (j, l)$. If we set $p = q^{-m}$ then any power of p is also a $2m + 1$ -th root of unity and $q = q^{-2m} = p^2$ so that our Hopf algebra is $c_p[SL_2]$ with its standard coquasitriangular structure with the correct factor $q^{m(m+1)} = p^{-m-1} = p^m = p^{-\frac{1}{2}}$. \square

We now recall explicitly that for q a primitive n -th root of unity and $q^2 \neq 1$, $u_q(sl_2)$ is generated by E, F, K , with relations, coproducts and coquasitriangular structure

$$E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-2}E, \quad KFK^{-1} = q^2F, \quad [E, F] = K - K^{-1},$$

$$\Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E,$$

$$\mathcal{R} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-2ab}}{[r]_{q^{-2}}!} F^r K^a \otimes E^r K^b,$$

where in our conventions we do not divide by the usual $q - q^{-1}$ in the $[E, F]$ -relation (and where we use q^{-2} rather than q^2 in the remaining relations compared with [15]). One can consider this as an unconventional normalisation of E which is cleaner when we are not interested in a classical limit. It gives a commutative Hopf algebra $u_{-1}(sl_2)$ when $q = -1$. We first show that double bosonisation gives us some version of such reduced quantum groups, agreeing for primitive odd roots. This was outlined in [15, Example 17.6] in the odd root case but we give a short derivation for all roots.

Lemma 4.2. [15] *Let q be a primitive n -th root of unity and let $H = \mathbb{C}_q \mathbb{Z}_n = \mathbb{C}_q[K]/(K^n - 1)$ be a quasitriangular Hopf algebra by $\mathcal{R}_K = \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} K^a \otimes K^b$ as in [14]. Let $B = \mathbb{C}[E]/(E^n)$ be a braided group in \mathcal{M}_H and dual $B^* = \mathbb{C}[F]/(F^n)$ in ${}_H\mathcal{M}$ with actions $E \triangleleft K = qE$ and $K \triangleright F = qF$.*

- (1) *The double bosonisation $B^{*\text{cop}} \bowtie H \bowtie B$ is a quasitriangular Hopf algebra, which we denote $\mathfrak{u}_q(sl_2)$, with the same coalgebra structure as above but with*

$$E^n = F^n = 0, \quad K^n = 1, \quad KEK^{-1} = q^{-1}E, \quad KFK^{-1} = qF, \quad [E, F] = K - K^{-1},$$

$$\mathcal{R}_{\mathfrak{u}_q(sl_2)} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-ab}}{[r]_{q^{-1}}!} F^r K^a \otimes E^r K^b.$$

- (2) *If $n = 2m + 1$ then $\mathfrak{u}_q(sl_2)$ is isomorphic to $u_{q^{-m}}(sl_2)$ with its standard quasitriangular structure.*

Proof. Here $EK \equiv (1 \otimes E)(K \otimes 1) = K \otimes E \triangleleft K = K \otimes qE \equiv qKE$ and $KF \equiv (1 \otimes K)(F \otimes 1) = K \triangleright F \otimes K = qF \otimes K \equiv qFK$. From the cross relations stated in Theorem 2.3, we also have

$$\begin{aligned} EF &= FE + \frac{1}{n} \sum_{a,b=0}^{n-1} q^{-ab} K^b \langle F, E \triangleleft K^a \rangle - \frac{1}{n} \sum_{a,b=0}^{n-1} q^{ab} K^a \langle K^b \triangleright F, E \rangle \\ &= FE + \frac{1}{n} \sum_{b=0}^{n-1} \left(\frac{1 - q^{-n(b-1)}}{1 - q^{-(b-1)}} \right) K^b \langle F, E \rangle - \frac{1}{n} \sum_{a=0}^{n-1} \left(\frac{1 - q^{n(a+1)}}{1 - q^{a+1}} \right) K^a \langle F, E \rangle \\ &= FE + K - K^{-1}, \end{aligned}$$

where we choose $\langle F, E \rangle = 1$. This is the same choice of normalisation for the braided line duality as in the calculation in Theorem 4.1. For the coproduct, clearly $\Delta K = K \otimes K$ while $\Delta E \equiv \Delta(1 \otimes E) = 1 \otimes 1 \otimes 1 \otimes E + 1 \otimes E \triangleleft \mathcal{R}_K^{(1)} \otimes \mathcal{R}_K^{(2)} \otimes 1 = 1 \otimes 1 \otimes 1 \otimes E + 1 \otimes E \otimes K \otimes 1 \equiv 1 \otimes E + E \otimes K$ and $\Delta F \equiv \Delta(F \otimes 1) = F \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \mathcal{R}_K^{-(1)} \otimes \mathcal{R}_K^{-(2)} \triangleright F \otimes 1 = F \otimes 1 \otimes 1 \otimes 1 + 1 \otimes K^{-1} \otimes F \otimes 1 \equiv F \otimes 1 + K^{-1} \otimes F$. Hence we have the relations and coalgebra as stated. Also from Theorem 2.3,

$$\mathcal{R}_{\mathfrak{u}_q(sl_2)} = \sum_{r=0}^{n-1} \left(\frac{F^r}{[r]_q!} \otimes \underline{S} E^r \right) \mathcal{R}_K = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{\frac{r(r-1)}{2}} q^{-ab}}{[r]_q!} F^r K^a \otimes E^r K^b,$$

which we write as stated. When $n = 2m + 1$, it is easy to see that the relations and quasitriangular structure become those of $u_p(sl_2)$ with $p = q^{-m}$, which are the same as in [15] after allowing for the normalisation of the generators. Note that if q is an even root of unity then $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ need not be factorisable, see Example 4.4. In fact, $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ is factorisable iff n is odd, which can be proven in a similar way to the proof in [8]. \square

We see that the double bosonisation $\mathfrak{u}_q(sl_2)$ recovers $u_p(sl_2)$ in the odd root of unity case with $p = q^{\frac{1}{2}}$, in line with the generic q case in [10]. Clearly $\mathfrak{u}_q(sl_2)$ has a PBW-type basis $\{F^i K^j E^k\}_{0 \leq i,j,k \leq n-1}$ as familiar in the odd case for $u_p(sl_2)$.

Corollary 4.3. *The basis $\{X^i t^j Y^k\}_{0 \leq i, j, k \leq n-1}$ of $\mathfrak{c}_q[SL_2]$ is, up to normalisation, dual to the PBW basis of $\mathfrak{u}_q(sl_2)$ in the sense*

$$\langle X^i t^j Y^k, F^{i'} K^{j'} E^{k'} \rangle = \delta_{i,i'} \delta_{k,k'} q^{jj'} [i]_{q^{-1}}! [k]_q!.$$

More precisely, $\left\{ \frac{X^i \delta_j(t) Y^k}{[i]_{q^{-1}}! [k]_q!} \right\}_{0 \leq i, j, k < n}$ is a dual basis to $\{F^i K^j E^k\}_{0 \leq i, j, k < n}$, where

$$\delta_j(t) = \frac{1}{n} \sum_{l=0}^{n-1} q^{-jl} t^l.$$

Proof. The duality pairing between the double and co-double bosonisations is

$$\langle X^i t^j Y^k, F^{i'} K^{j'} E^{k'} \rangle = \langle X^{i(\text{op})}, F^{i'} \rangle \langle t^j, K^{j'} \rangle \langle Y^k, E^{k'} \rangle,$$

where the pairing between $(\mathbb{C}[X]/(X^n))^{\text{op}}$ and $(\mathbb{C}[F]/(F^n))^{\text{cop}}$ implied by $\langle X, F \rangle = 1$ is $\langle X^{i(\text{op})}, F^{i'} \rangle = \delta_{i,i'} [i]_{q^{-1}}!$ while $\langle t^j, K^{j'} \rangle = q^{jj'}$ is implied by $\langle t, K \rangle = q$. The latter is the duality pairing in the Pontryagin sense in which \mathbb{Z}_n is self-dual, and can be written as a usual dual pairing with the δ_j . Equally well, $\left\{ \frac{F^i \delta_j(K) E^k}{[i]_{q^{-1}}! [k]_q!} \right\}_{0 \leq i, j, k < n}$ is a dual basis to $\{X^i t^j Y^k\}_{0 \leq i, j, k < n}$. \square

This applies even when $q = -1$, in that case as a self-duality pairing.

Example 4.4. If $q = -1$ then the double bosonisation $\mathfrak{u}_{-1}(sl_2) = B^{*\text{cop}} \bowtie H \bowtie B$ from Lemma 4.2 has relations and coalgebra structure given by

$$\begin{aligned} E^2 = F^2 = 0, \quad K^2 = 1, \quad EF = FE, \quad KE = -EK, \quad KF = -FK, \\ \Delta K = K \otimes K, \quad \Delta F = F \otimes 1 + K \otimes F, \quad \Delta E = E \otimes K + 1 \otimes E \end{aligned}$$

and is self-dual and strictly quasitriangular with

$$\mathcal{R} = (1 \otimes 1 - F \otimes E) \mathcal{R}_K, \quad \mathcal{R}_K = \frac{1}{2} (1 \otimes 1 + 1 \otimes K + K \otimes 1 - K \otimes K).$$

It is easy to check that this is not triangular, i.e. $Q := \mathcal{R}_{21} \mathcal{R} = 1 \otimes 1 - E \otimes F - KF \otimes EK - EKF \otimes FKE \neq 1 \otimes 1$, and also not factorisable in the sense that the map $\mathfrak{u}_{-1}(sl_2)^* \rightarrow \mathfrak{u}_{-1}(sl_2)$ which sends $\phi \mapsto (\phi \otimes \text{id})Q$ is not surjective (the element $FK \in \mathfrak{u}_{-1}(sl_2)$ is not in the image). On the other hand, Theorem 4.1 (1) gives us an isomorphic Hopf algebra by $X \mapsto F, Y \mapsto E$ and $t \mapsto K$, so our Hopf algebra is self-dual, i.e., $\mathfrak{u}_{-1}(sl_2) \cong \mathfrak{c}_{-1}[SL_2]$. Note that $\mathfrak{u}_{-1}(sl_2)$ has the same dimension and the coalgebra structure of $u_{-1}(sl_2)$ but cannot be isomorphic, being noncommutative. One can also check that $\mathfrak{c}_{-1}[SL_2]$ is not isomorphic as a Hopf algebra to $c_{-1}[SL_2]$ and the latter, being noncocommutative, cannot be dual to $u_{-1}(sl_2)$.

5. APPLICATION TO HOPF ALGEBRA FOURIER TRANSFORM

As a corollary of the above results, we briefly consider Hopf algebra Fourier transform between our double and co-double bosonisations. Recall from standard Hopf algebra theory, e.g. [14], that for a finite-dimensional Hopf algebra H there is, up to scale, a unique right integral structure $\int : H \rightarrow k$ satisfying

$$\left(\int \otimes \text{id} \right) \Delta h = \left(\int h \right) 1$$

for all $h \in H$. Such a right integral is the main ingredient for Fourier transform $\mathcal{F} : H \rightarrow H^*$. The following preliminary lemma is essentially well-known (see [14, Proposition, 1.7.7]), but for completeness we give the easier part that we need.

Lemma 5.1. *Let \int, \int^* be right integrals on finite-dimensional Hopf algebras H, H^* respectively and $\mu = \int(\int^*)$. The Fourier transform $\mathcal{F} : H \rightarrow H^*$ and adjunct $\mathcal{F}^* : H^* \rightarrow H$ obey*

$$\mathcal{F}(h) = \sum_a \left(\int e_a h \right) f^a, \quad \mathcal{F}^*(\phi) = \sum_a e_a \left(\int^* \phi f^a \right), \quad \mathcal{F}^* \circ \mathcal{F} = \mu S,$$

where $\{e_a\}$ is basis of H , $\{f^a\}$ is the dual basis of H^* . Hence \mathcal{F} is invertible if $\mu \neq 0$.

Proof. We write $\int^* = \Lambda^*$ when regarded as element in H . Then

$$\begin{aligned}\mathcal{F}^* \circ \mathcal{F}(h) &= e_a \left(\int^* \left(\int e_b h \right) f^b f^a \right) = \left(\int e_{a_{(1)}} h \right) e_{a_{(2)}} \left(\int^* f^a \right) \\ &= \left(\int \Lambda^*_{(1)} h_{(1)} \right) \Lambda^*_{(2)} h_{(2)} Sh_{(3)} = \left(\int \Lambda^* h_{(1)} \right) Sh_{(2)} = \left(\int \Lambda^* \right) Sh = \mu Sh.\end{aligned}$$

If $\mu \neq 0$ then this implies that \mathcal{F} is injective and hence in our case invertible (with a bit more work [14] one can show that the inverse is $\mu^{-1}S^{-1}\mathcal{F}^*$). \square

Proposition 5.2. *Let q be a primitive n -th root of unity. The Fourier transform $\mathcal{F} : \mathfrak{c}_q[SL_2] \rightarrow \mathfrak{u}_q(sl_2)$ is invertible and given by*

$$\mathcal{F}(X^\alpha t^\beta Y^\gamma) = \sum_{l=0}^{n-1} \frac{q^{-(l+\alpha)(1-\beta)+\beta(n-1-\gamma)}}{n[n-1-\alpha]_{q^{-1}}![n-1-\gamma]_q!} F^{n-1-\alpha} K^l E^{n-1-\gamma}.$$

Proof. The right integral for $\mathfrak{c}_q[SL_2]$ is given by

$$\int X^\alpha t^\beta Y^\gamma = \begin{cases} 1, & \text{if } \alpha = \gamma = n-1, \beta = 1 \\ 0, & \text{otherwise.} \end{cases}$$

This integral is equivalent in usual generators to $\int b^{n-1} c^{n-1} = 1$ and zero otherwise. We use Corollary 4.3 to give us the basis $\{e_a\} = \{X^i t^j Y^k\}_{0 \leq i,j,k \leq n-1}$ of $\mathfrak{c}_q[SL_2]$ and the dual basis $\{f^a\} = \{\frac{F^i \delta_j(K) E^k}{[i]_{q^{-1}}! [j]_q! [k]_q!}\}_{0 \leq i,j,k \leq n-1}$ of $\mathfrak{u}_q(sl_2)$. Then

$$\begin{aligned}\mathcal{F}(X^\alpha t^\beta Y^\gamma) &= \sum_{i,j,k=0}^{n-1} \left(\int X^i t^j Y^k X^\alpha t^\beta Y^\gamma \right) \frac{F^i \delta_j(K) E^k}{[i]_{q^{-1}}! [j]_q! [k]_q!} \\ &= \sum_{i,j,k=0}^{n-1} q^{-\alpha j + \beta k} \left(\int X^{i+\alpha} t^{j+\beta} Y^{k+\gamma} \right) \frac{F^i \delta_j(K) E^k}{[i]_{q^{-1}}! [j]_q! [k]_q!} \\ &= q^{-\alpha(1-\beta)+\beta(n-1-\gamma)} \frac{F^{2-\alpha} \delta_{1-\beta}(K) E^{n-1-\gamma}}{[n-1-\alpha]_{q^{-1}}! [n-1-\gamma]_q!} \\ &= \sum_{l=0}^{n-1} \frac{q^{-(l+\alpha)(1-\beta)+\beta(n-1-\gamma)}}{(n-1+1)[n-1-\alpha]_{q^{-1}}! [n-1-\gamma]_q!} F^{n-1-\alpha} K^l E^{n-1-\gamma}.\end{aligned}$$

The similar right integral of $\mathfrak{u}_q(sl_2)$ and resulting μ are

$$\int^* F^\alpha K^\beta E^\gamma = \begin{cases} 1 & \text{if } \alpha = \gamma = n-1, \beta = 1 \\ 0 & \text{otherwise,} \end{cases} \quad \mu = \frac{q^{-1}}{n[n-1]_{q^{-1}}! [n-1]_q!},$$

which is nonzero. \square

It appears to be a hard computational problem to give the general formula of the inverse Fourier transform, but one can compute it for specific cases.

Example 5.3. Let q be a primitive cube root of unity. First, observe that for $\alpha, \beta = 0, 1, 2$, we have

$$\begin{aligned}[E^\alpha, F^\beta] &= F^{\beta-1} ([\alpha]_q [\beta]_q K - [\alpha]_{q^{-1}} [\beta]_{q^{-1}} K^{-1}) E^{\alpha-1} \\ &\quad + F^{\beta-2} ([2]_q K - [2]_{q^{-1}} K^{-1}) (K - K^{-1}) E^{\alpha-2}\end{aligned}$$

in $\mathfrak{u}_q(sl_2)$. Using this relation, we obtain

$$\begin{aligned}\mathcal{F}^*(F^\alpha K^\beta E^\gamma) &= \sum_{l=0}^2 \frac{q^{\beta(2-\alpha)+(\gamma-l)(1-\beta)}}{3[2-\alpha]_{q^{-1}}! [2-\gamma]_q!} X^{2-\alpha} t^l Y^{2-\gamma} \\ &\quad + \sum_{l=0}^2 \frac{q^{\beta(l-\alpha-\gamma)} ([\gamma]_q [3-\alpha]_q - q^{2(\gamma-l)+1} [\gamma]_{q^{-1}} [3-\alpha]_{q^{-1}})}{3[3-\alpha]_{q^{-1}}! [3-\gamma]_q!} F^{3-\alpha} t^l Y^{3-\gamma}\end{aligned}$$

$$- \frac{q^{2\beta+1}}{3[4-\alpha]_{q^{-1}}![4-\gamma]_q!} X^{4-\alpha} t^2 Y^{4-\gamma}.$$

One can check that $\mathcal{F}^* \mathcal{F}(X^\alpha t^\beta Y^\gamma) = \mu S(X^\alpha t^\beta Y^\gamma)$, where $\mu = \frac{q^{-1}}{3[2]_{q^{-1}}![2]_q!} = \frac{q^2}{3}$ and

$$\begin{aligned} S(X^\alpha t^\beta Y^\gamma) &= \frac{q^{\alpha\beta-\beta\gamma}}{[2-\alpha]_{q^{-1}}![\alpha]_{q^{-1}}![2-\gamma]_q![\gamma]_q!} X^\alpha t^{-\delta} Y^\gamma \\ &+ \frac{q^{\alpha\beta-\beta\gamma}(q^{2\beta-2} - q^{\delta-2})}{[2-\alpha]_{q^{-1}}![\alpha]_{q^{-1}}![1-\gamma]_q![1+\gamma]_q!} X^{\alpha+1} t^{-\delta-1} Y^{\gamma+1} \\ &- \frac{1 + q^{\beta+1} + q^{2\beta+2}}{[2-\alpha]_{q^{-1}}![2+\alpha]_{q^{-1}}![2-\gamma]_q![2+\gamma]_q!} X^{2+\alpha} t^2 Y^{2+\gamma}, \end{aligned}$$

where $\delta = \alpha + \beta + \gamma$.

Example 5.4. At $q = -1$, the Fourier transform in Proposition 5.2 combined with the self-duality in Example 4.4 becomes a Fourier transform operator $\mathfrak{c}_{-1}[SL_2] \rightarrow \mathfrak{c}_{-1}[SL_2]$. This has eigenvalues $\pm \frac{i}{\sqrt{2}}$ with multiplicity 2, $\pm \frac{(-1)^{1/4}}{\sqrt{2}}$ and $\pm \frac{(-1)^{3/4}}{\sqrt{2}}$ with multiplicity 1, and characteristic polynomial $f(x) = \frac{1}{16} + \frac{x^2}{4} + \frac{x^4}{2} + x^6 + x^8$. We also have

$$\mathcal{F}^*(F^a K^b E^c) = \frac{1}{2} \sum_{l=0}^1 (-1)^{(1-b)(c-l)+b(1-a)} X^{1-a} t^l Y^{1-c}$$

and one can check that $\mathcal{F}^{-1} = \mu^{-1} S^{-1} \mathcal{F}^*$ as in Lemma 5.1.

It is known that Fourier transform behaves well with respect to the coregular representation. This implies that it behaves well with respect to any covariant calculus. Thus, let (Ω^1, d) be a left-covariant calculus on H . By definition, a differential calculus means an H - H -bimodule Ω^1 together with a derivation $d : H \rightarrow \Omega^1$ such that the map $H \otimes H \rightarrow \Omega^1$ sending $h \otimes g \mapsto hdg$ is surjective. This is left covariant if the map

$$\Delta_L(hdg) = h_{(1)}g_{(1)} \otimes h_{(2)}dg_{(2)}, \quad \Delta_L : \Omega^1 \rightarrow H \otimes \Omega^1$$

is well-defined. In this case it is a left coaction and d is a comodule map with respect to the left coproduct on H . By the Hopf-module lemma, such Ω^1 are free modules over their space Λ^1 of invariant 1-forms while $dh = h_{(1)}\varpi\pi_\epsilon h_{(2)}$ for all $h \in H$, where $\pi_\epsilon = \text{id} - 1\epsilon : H \rightarrow H^+$ and $\varpi : H^+ \rightarrow \Lambda^1$ is the Maurer-Cartan form $\varpi(h) = Sh_{(1)}dh_{(2)}$ for all $h \in H^+$. We refer to [23, 15] for details. The following is known, see e.g. [16], but we include a short derivation in our conventions. In our case H is finite-dimensional.

Lemma 5.5. *Let $\{e_a\}$ be a basis of Λ^1 , $\{f^a\}$ a dual basis and define partial derivatives $\partial^a : H \rightarrow H$ by $dh = \sum_a (\partial^a h) e_a$ and $\chi_a \in H^*$ by $\chi_a(h) = \langle f^a, \varpi\pi_\epsilon S^{-1}h \rangle$ for all $h \in H$. Then $\mathcal{F}(\partial^a h) = (\mathcal{F}h)\chi_a$ for all $h \in H$.*

Proof. Using the right-integral property, we have

$$\begin{aligned} \mathcal{F}(\partial^a h) &= \mathcal{F}(h_{(1)}) \langle f^a, \varpi\pi_\epsilon h_{(2)} \rangle = \sum_b \left(\int e_{b(1)} h_{(1)} \right) f^b \langle f^a, \varpi\pi_\epsilon ((S^{-1}e_{b(3)})e_{b(2)}h_{(2)}) \rangle \\ &= \sum_b \left(\int e_{b(1)} h \right) f^b \langle f^a, \varpi\pi_\epsilon (S^{-1}e_{b(2)}) \rangle = \sum_{b,c} \left(\int e_b h \right) f^b f^c \langle f^a, \varpi\pi_\epsilon (S^{-1}e_c) \rangle = (\mathcal{F}h)\chi_a. \end{aligned}$$

□

Example 5.6. The 3D calculus c.f. [23] has left-invariant basic 1-forms e_\pm, e_0 with $e_\pm h = p^{|h|} h e_\pm$ and $e_0 h = p^{2|h|} h e_0$ where $p = q^{-m}$ and $| \cdot |$ denotes a grading with a, c grade 1 and b, d grade -1 as a \mathbb{Z}_n -grading of $c_p[SL_2]$. Correspondingly for $\mathfrak{c}_q[SL_2]$, we have a calculus with $|X| = 0, |t| = |Y| = 2$ and one can compute

$$\begin{aligned} dX &= q^{-m} t e_-, \quad dt = (1+q)(q(q^{-m} - q^m)tY e_- + t e_0), \\ dY &= (q^{-1} - 1)^{-1} e_+ + (1+q)Y e_0 + q(q^{-m} - q^m)Y^2 e_-, \end{aligned}$$

which implies on a general monomial basis element that

$$\begin{aligned} d(X^i t^j Y^k) = & (q^{-1} - 1)^{-1} [k]_q X^i t^j Y^{k-1} e_+ + (1 + q) [j + k]_{q^2} X^i t^j Y^k e_0 \\ & + \left(q(q^{-m} - q^m) [2j + k]_q X^i t^j Y^{k+1} + [i]_{q^{-1}} q^{-m+j+k} X^{i-1} t^{j+1} Y^k \right) e_-. \end{aligned}$$

We determine $\chi_a \in u_p(sl_2)$ from $\langle X^i t^j Y^k, \chi_a \rangle = \epsilon(\partial^a(X^i t^j Y^k))$ with the result

$$\begin{aligned} \chi_+ &= \sum_{j=0}^{n-1} \frac{\delta_j(K) E}{q^{-1} - 1} = \sum_{i,j=0}^{n-1} \frac{q^{-ij}}{n(q^{-1} - 1)} K^i E = \frac{E}{q^{-1} - 1}, \\ \chi_0 &= \sum_{j=0}^{n-1} (1 + q) [j]_{q^2} \delta_j(K) = \sum_{i,j=0}^{n-1} \frac{q^{-ij} (1 - q^{2j})}{n(1 - q)} K^i = \frac{1 - K^2}{1 - q}, \\ \chi_- &= \sum_{j=0}^{n-1} q^{-m+j} F \delta_j(K) = \sum_{i,j=0}^{n-1} \frac{q^{-m} q^{j-ij}}{n} F K^i = q^{-m} F K. \end{aligned}$$

These are versions of similar elements found for $\mathbb{C}_q[SU_2]$ with real q in [23].

6. CONSTRUCTION OF $u_q(sl_3)$ AND $c_q[SL_3]$ BY (CO)DOUBLE BOSONISATION

As mentioned in the introduction, double bosonisation can in principle be used iteratively to construct all the $u_q(\mathfrak{g})$ [11, 15] and hence now, by making the corresponding co-double bosonisation at each step, an appropriate dual $c_q[G]$. The quantum-braided planes and their duals adjoined at each step generally have a more straightforward duality pairing given by braided factorial operators, see [16]. Here we find

$$u_q(sl_3) = \mathfrak{c}_q^2 \rtimes \widetilde{u_q(sl_2)} \ltimes \mathfrak{c}_q^2 = \mathfrak{c}_q^2 \rtimes (\mathfrak{c}_q^1 \rtimes \mathbb{C}\mathbb{Z}_n^2 \ltimes \mathfrak{c}_q^1) \ltimes \mathfrak{c}_q^2$$

for certain n -th roots of unity, and a parallel result for $c_q[SL_3]$. The former was explained for generic q in [10] but at roots of unity we need to be much more careful.

6.1. Construction of $u_q(sl_3)$ from $u_q(sl_2)$. The quantum group $u_q(sl_3)$ in more or less standard conventions is generated by E_i, F_i, K_i for $i = 1, 2$, with, c.f. [4],

$$E_i^n = F_i^n = 0, \quad K_i^n = 1,$$

$$K_i K_j = K_j K_i, \quad E_i K_j = q^{a_{ij}} K_i E_j, \quad K_i F_j = q^{a_{ij}} F_j K_i, \quad [E_i, F_j] = \delta_{ij} (K_i - K_i^{-1}),$$

$$\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

where $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$. As before, we absorbed a factor $q - q^{-1}$ in the cross relation as a normalisation of E_i . We also require the q -Serre relations

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0$$

for $i \neq j$. Note that $u_q(sl_2)$ appears as a sub-Hopf algebra generated by E_1, F_1, K_1 .

Let q be a primitive n -th root of unity with $n = 2m + 1$ and $p = q^{-m} = q^{\frac{1}{2}}$. Let $B = \mathfrak{c}_q^2$ be the algebra generated by e_1, e_2 with relation $e_2 e_1 = q^{-m} e_1 e_2$ in the category of right $u_q(sl_2)$ -modules. The canonical left-action of $u_q(sl_2)$ on B is given by

$$\begin{aligned} (6.1) \quad \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft K &= \langle K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} q^{-m} & 0 \\ 0 & q^m \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft E &= \langle E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \\ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \triangleleft F &= \langle F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \end{aligned}$$

where $\lambda = q^m - q^{-m}$. The duality between $u_q(sl_2)$ and $c_{q^{-m}}[SL_2]$ is the standard one when the former is identified with $u_{q^{-m}}(sl_2)$, or can be obtained from Corollary 4.3.

Lemma 6.1. *Let q be a primitive n -th root of 1 with $n = 2m + 1$ such that $\beta^2 = 3$ has a solution mod n . Let $H = \widetilde{\mathfrak{u}_q(sl_2)} = \mathfrak{u}_q(sl_2) \otimes \mathbb{C}_q[g]/(g^n - 1)$, and let g act on e_i by*

$$e_i \triangleleft g = q^{m\beta} e_i.$$

Then \mathfrak{c}_q^2 is a braided-Hopf algebra in the braided category of right H -modules with

$$\begin{aligned} e_1^n = e_2^n = 0, \quad e_2 e_1 = q^{-m} e_1 e_2, \quad \underline{\Delta}(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \underline{\epsilon}(e_i) = 0, \quad \underline{S}(e_i) = -e_i, \\ \Psi(e_i \otimes e_i) = q e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = q^{-m} e_2 \otimes e_1, \quad \Psi(e_2 \otimes e_1) = q^{-m} e_1 \otimes e_2 + (q-1) e_2 \otimes e_1. \end{aligned}$$

Proof. The quasitriangular structure of $\widetilde{\mathfrak{u}_q(sl_2)}$ is given by $\mathcal{R}_{\mathfrak{u}_q(sl_2)} \mathcal{R}_g$, where $\mathcal{R}_g = \frac{1}{n} \sum_{s,t=0}^{n-1} q^{-st} g^s \otimes g^t$ and $\mathcal{R}_{\mathfrak{u}_q(sl_2)}$ is given in Lemma 4.2. Thus, we can compute that

$$\Psi(e_i \otimes e_j) = q^{m^2 \beta^2} (e_i \otimes e_j) \triangleleft \mathcal{R}_{\mathfrak{u}_q(sl_2)}.$$

This braiding is equal to the correctly normalised braiding in the statement (as needed for $\underline{\Delta}$ to extend as a homomorphism to the braided tensor product algebra) iff $m^2 \beta^2 = m(m-1) \bmod n$, or $m\beta^2 = m-1$ since any $m > 0$ is invertible mod n (this is true for $m = 1$ and if $m > 1$ then m and $2m+1$ are coprime). Thus the condition for \mathfrak{c}_q^2 to form a braided-Hopf algebra in the category of $\widetilde{\mathfrak{u}_q(sl_2)}$ -modules by an action of the stated form is $m(\beta^2 - 1) = -1 = 2m \bmod n$, or $\beta^2 = 3 \bmod n$. Some version of this lemma was largely in [1], working directly with $p = q^{-m}$. \square

Here $\beta = 0$ is only possible for $m = 1$, i.e., $n = 3$. In this case \mathfrak{c}_q^2 is already a braided-Hopf algebra in the category of $\mathfrak{u}_q(sl_2)$ -modules without a central extension being needed. Otherwise, the least n satisfying the condition is $n = 11$ with $\beta = 5$. For n prime, β exists if and only if $n = \pm 1 \bmod 12$, see [22].

The dual $B^* = (\mathfrak{c}_q^2)^* \in {}_H M$ is generated by f_1, f_2 satisfying the same relations $f_2 f_1 = q^{-m} f_1 f_2$ and additive braided coproduct as B but with the left action

$$\begin{aligned} (6.2) \quad K \triangleright (f_1 \ f_2) &= (f_1 \ f_2) \begin{pmatrix} q^{-m} & 0 \\ 0 & q^m \end{pmatrix}, \quad g \triangleright f_i = q^{m\beta} f_i, \\ E \triangleright (f_1 \ f_2) &= (f_1 \ f_2) \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \quad F \triangleright (f_1 \ f_2) = (f_1 \ f_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Lemma 6.2. *The quantum-braided planes \mathfrak{c}_q^2 and $(\mathfrak{c}_q^2)^*$ in Lemma 6.1 are dually paired by $\langle e_1^r e_2^s, f_1^{r'} f_2^{s'} \rangle = \delta_{r,r'} \delta_{s,s'} [r]_q! [s]_q!$.*

Proof. It is not hard to see that $\langle e_i^r, f_i^{r'} \rangle = \delta_{r,r'} [r]_q!$ and this implies that

$$\begin{aligned} \langle e_1^r e_2^s, f_1^{r'} f_2^{s'} \rangle &= \langle e_1^r \otimes e_2^s, \sum_{r_1=0}^{r'} \sum_{s_1=0}^{s'} \begin{bmatrix} r' \\ r_1 \end{bmatrix}_q \begin{bmatrix} s' \\ s_1 \end{bmatrix}_q q^{-ms_1(r'-r_1)} f_1^{r_1} f_2^{s_1} \otimes f_1^{r'-r_1} f_2^{s'-s_1} \rangle \\ &= \langle e_1^r \otimes e_2^s, f_1^{r'} \otimes f_2^{s'} \rangle = \delta_{r,r'} \delta_{s,s'} [r]_q! [s]_q!. \end{aligned}$$

\square

In the double bosonisation, we read the generators e_1, e_2 of the quantum-braided plane $B = \mathfrak{c}_q^2$ as E_{12} and E_2 respectively. Similarly, the generators f_1, f_2 of its dual quantum-braided plane $(\mathfrak{c}_q^2)^*$ are read as F_{12}, F_2 respectively. Also, we read the generators E, F, K of $\mathfrak{u}_q(sl_2)$ as E_1, F_1 and K_1 so that

$$E_1^n = F_1^n = 0, \quad K_1^n = 1, \quad K_1 E_1 K_1^{-1} = q^{-1} E_1, \quad K_1 F_1 K_1^{-1} = q F_1, \quad [E_1, F_1] = K_1 - K_1^{-1}.$$

Lemma 6.3. *Suppose the setting of Lemma 6.1 with n odd and $\beta^2 = 3$ solved mod n .*

- (1) *The double bosonisation of \mathfrak{c}_q^2 , which we denote $\mathfrak{u}_q(sl_3)$, is generated by E_i, F_i, K_1, g for $i = 1, 2$, with E_1, F_1, K_1 generating $\mathfrak{u}_q(sl_2)$ as a sub-Hopf algebra, and*

$$\begin{aligned} E_2 K_1 &= q^m K_1 E_2, \quad E_2 g = q^{m\beta} g E_2, \quad K_1 F_2 = q^m F_2 K_1, \quad g F_2 = q^{m\beta} F_2 g, \\ [E_1, F_2] &= [E_2, F_1] = 0, \quad [E_2, F_2] = K_1^m g^{m\beta} - K_1^{-m} g^{-m\beta}, \end{aligned}$$

$$\begin{aligned}
\{E_i^2, E_j\} &= (q^m + q^{-m})E_i E_j E_i, \quad \{F_i^2, F_j\} = (q^m + q^{-m})F_i F_j F_i; \quad i \neq j, \\
\Delta E_2 &= 1 \otimes E_2 + E_2 \otimes K_1^m g^{m\beta}, \quad \Delta F_2 = F_2 \otimes 1 + g^{-m\beta} K_1^{-m} \otimes F_2, \\
\mathcal{R}_{u_q(sl_3)} &= \frac{1}{n^2} \sum \frac{(-1)^{r+v+w} q^{vw-st-ab}}{[r]_{q^{-1}}! [v]_{q^{-1}}! [w]_{q^{-1}}!} F_{12}^v F_2^w F_1^r K_1^s g^a \otimes E_{12}^v E_2^w E_1^r K_1^t g^b, \\
&\text{where we sum over } r, s, a, t, b, v, w \text{ from } 0 \text{ to } n-1 \text{ and} \\
E_2 E_1 &= q^m E_1 E_2 + \lambda E_{12}, \quad F_1 F_2 = q^{-m} F_2 F_1 + F_{12}; \quad \lambda = q^m - q^{-m}.
\end{aligned}$$

(2) If $n > 3$ and is not divisible by 3 then $u_q(sl_3)$ is isomorphic to $u_{q^{-m}}(sl_3)$.

Proof. (1) This is a direct computation using Theorem 2.3. First, we have that $E_2 h = h_{(1)}(E_2 \triangleleft h_{(2)})$ and $h F_2 = (h_{(1)} \triangleright F_2) h_{(2)}$ for all $h \in u_q(sl_2)$ and using the correct actions mentioned above. Those not involving E_{12}, F_{12} are as listed, while two more are regarded in the statement as definitions of E_{12}, F_{12} in terms of the other generators. In this case the remaining cross relations

$$E_{12} K_1 = q^{-m} K_1 E_{12}, \quad K_1 F_{12} = q^{-m} F_{12} K_1, \quad E_{12} g = q^{m\beta} g E_{12}, \quad g F_{12} = q^{m\beta} F_{12} g$$

are all empty and can be dropped. Similarly, the first two of

$$[E_{12}, F_1] = K_1^{-1} E_2, \quad [E_1, F_{12}] = \lambda F_2 K_1, \quad E_{12} E_1 = q^{-m} E_1 E_{12}, \quad F_1 F_{12} = q^m F_{12} F_1$$

are empty and can be dropped. The remaining two and the original quantum-braided plane relations $E_{12} E_2 = q^m E_2 E_{12}, F_{12} F_2 = q^m F_2 F_{12}$ are the four q -Serre relations stated for $i \neq j$. We next look at the cross relations between the two quantum-braided planes. For example,

$$[E_2, F_2] = \mathcal{R}^{(2)} \langle F_2, E_2 \triangleleft \mathcal{R}^{(1)} \rangle - \mathcal{R}^{-(1)} \langle \mathcal{R}^{-(2)} \triangleright F_2, E_2 \rangle.$$

Putting in the form of \mathcal{R} and \mathcal{R}^{-1} gives the stated cross relation. One similarly has

$$[E_{12}, F_2] = -E_1 K_1^m g^{m\beta}, \quad [E_2, F_{12}] = \lambda g^{-m\beta} K_1^{-m} F_1, \quad [E_{12}, F_{12}] = K_1^{-m} g^{m\beta} - K_1^m g^{-m\beta}$$

of which the first two are empty by a similar computation to the one above and the last is also empty by a more complicated calculation. In fact all these identities can be useful even though we do not include them in the defining relations. We also have

$$\begin{aligned}
\Delta E_2 &= 1 \otimes E_2 + \frac{1}{n^2} \sum \frac{(-1)^r}{[r]_{q^{-1}}!} q^{-st-ab} (E_2 \triangleleft F_1^r K_1^s g^a) \otimes E_1^r K_1^t g^b \\
&= 1 \otimes E_2 + \frac{1}{n^2} \sum q^{-s(t-m)-a(b-m\beta)} E_2 \otimes K_1^t g^b = 1 \otimes E_2 + E_2 \otimes K_1^m g^{m\beta},
\end{aligned}$$

where we sum over r, s, a, t, b from 0 to $n-1$. To compute ΔF_2 , we need

$$\mathcal{R}^{-1} = S \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} = \frac{1}{n^2} \sum \frac{q^{-st-ab}}{[r]_{q^{-1}}!} g^{-a} K_1^{r-s} F_1^r \otimes E_1^r K_1^t g^b.$$

Only the first term contributes when acting on F_2 ,

$$\begin{aligned}
\Delta F_2 &= F_2 \otimes 1 + \frac{1}{n^2} \sum \frac{(-1)^r}{[r]_{q^{-1}}!} q^{-st-ab} g^{-a} K_1^{r-s} F_1^r \otimes E_1^r K_1^t g^b \triangleright F_2 \\
&= F_2 \otimes 1 + \frac{1}{n^2} \sum q^{-t(s-m)-b(a-m\beta)} g^{-a} K_1^{-s} \otimes F_2 = F_2 \otimes 1 + g^{-m\beta} K_1^{-m} \otimes F_2
\end{aligned}$$

and similarly for ΔE_2 . One also has

$$\Delta E_{12} = 1 \otimes E_{12} + E_{12} \otimes K_1^{-m} g^{m\beta} - E_2 \otimes E_1 K_1^m g^{m\beta},$$

$$\Delta F_{12} = F_{12} \otimes 1 + g^{-m\beta} K_1^m \otimes F_{12} + \lambda g^{-m\beta} K_1^{-m} F \otimes F_2$$

which we did not state as E_{12}, F_{12} are not generators. By Theorem 2.3 and Lemma 6.2, the quasitriangular structure of $u_q(sl_3)$ is

$$(6.3) \quad \mathcal{R}_{u_q(sl_3)} = \left(\sum_{v,w=0}^{n-1} \frac{F_{12}^v F_2^w}{[v]_{q^{-1}}! [w]_{q^{-1}}!} \otimes \underline{S}(E_{12}^v E_2^w) \right) \mathcal{R}_{u_q(sl_2)} \mathcal{R}_g,$$

where $\mathcal{R}_{u_q(sl_2)}\mathcal{R}_g$ is explained in the proof of Lemma 6.1. By (2.5) for the braided-antipode, we find

$$\underline{S}(E_{12}^v E_2^w) = (-1)^{v+w} q^{\frac{v(v-1)+w(w-1)}{2} + vw} E_{12}^w E_2^v,$$

so that (6.3) becomes the expression stated.

(2) If $m > 1$, we define $\varphi : u_{q^{-m}}(sl_3) \rightarrow u_q(sl_3)$ by

$$\varphi(E_i) = E_i, \quad \varphi(F_i) = F_i, \quad \varphi(K_1) = K_1, \quad \varphi(K_2) = K_1^m g^{m\beta}.$$

It is easy to see that φ is an algebra and coalgebra map. In the other direction, when $m > 1$, β is invertible mod n iff 3 is. We then define $\phi : u_q(sl_3) \rightarrow u_{q^{-m}}(sl_3)$ by

$$\phi(E_i) = E_i, \quad \phi(F_i) = F_i, \quad \phi(K_1) = K_1, \quad \phi(g) = (K_1^{-m} K_2)^{\frac{1}{m\beta}},$$

which is clearly inverse to φ . \square

We again write $p = q^{-m}$ so that $u_q(sl_3)$ is isomorphic to $u_p(sl_3)$ under our assumptions, where $n = 33$ and $\beta = 6$ is the first case excluded. The double bosonisation construction also gives $\{F_{12}^{i_1} F_2^{i_2} F_1^{i_3} K_1^{i_4} g^{i_5} E_1^{i_6} E_{12}^{i_7} E_2^{i_8}\}$ as a basis of $u_q(sl_3)$.

Example 6.4. As mentioned before, when q is a primitive cubic root of unity i.e., when $\beta = 0$, \mathfrak{c}_q^2 is already a braided-Hopf algebra in the category of $u_q(sl_2)$ -modules without an extension needed. Then Theorem 2.3 gives us a quasitriangular Hopf algebra, which we denote $u'_q(sl_3)$, generated by E_i, F_i, K_1 with $i = 1, 2$ with the relations and coproducts

$$\begin{aligned} E_i K_1 &= q K_1 E_i, \quad K_1 F_i = q F_i K_1, \quad [E_i, F_j] = \delta_{i,j} (K_1 - K_1^{-1}), \\ \{E_i^2, E_j\} &= (q + q^{-1}) E_i E_j E_i, \quad \{F_i^2, F_j\} = (q + q^{-1}) F_i F_j F_i; \quad i \neq j, \\ \Delta E_i &= 1 \otimes E_i + E_i \otimes K_1, \quad \Delta F_i = F_i \otimes 1 + K_1^{-1} \otimes F_i, \\ \mathcal{R}_H &= \frac{1}{9} \sum \frac{(-1)^{r+v+w} q^{vw-st}}{[r]_{q^{-1}}! [v]_{q^{-1}}! [w]_{q^{-1}}!} F_{12}^v F_2^w F_1^r K_1^s \otimes E_{12}^v E_2^w E_1^r K_1^t, \end{aligned}$$

where the sum is over r, s, t, v, w from 0 to 2. This $u'_q(sl_3)$ is not isomorphic to $u_{q^{-1}}(sl_3)$ since we do not have the generator K_2 . However, the element $K_1^{-1} K_2$ is central and group-like in $u_{q^{-1}}(sl_3)$ and $u'_q(sl_3) \cong u_{q^{-1}}(sl_3) / \langle K_1^{-1} K_2 - 1 \rangle$. In addition, Lemma 6.3 still applies and g is already group-like, and central when $\beta = 0$. Therefore we have $u_q(sl_3) = u'_q(sl_3) \otimes \mathbb{C}_q[g] / (g^3 - 1)$ for $m = 1$.

6.2. Construction of $\mathfrak{c}_q[SL_3]$ from $\mathfrak{c}_q[SL_2]$. Recall, see e.g. [14], that the coquasitriangular Hopf algebra $\mathbb{C}_q[SL_3]$ is generated by $\mathbf{t} = (t^i_j)$ for $i, j = 1, 2, 3$, with matrix-form of coproduct $\Delta \mathbf{t} = \mathbf{t} \otimes \mathbf{t}$, and for $i < k, j < l$, the relations

$$[t^i_l, t^j_j]_q = 0, \quad [t^k_j, t^i_j]_q = 0, \quad [t^i_l, t^k_j] = 0, \quad [t^k_l, t^i_l] = \lambda t^i_l t^k_j,$$

$$\det_q(\mathbf{t}) := t^1_1(t^2_2 t^3_3 - q^{-1} t^2_3 t^3_2) - q^{-1} t^1_2(t^2_1 t^3_3 - q^{-1} t^2_3 t^3_1) + q^{-2} t^1_3(t^2_1 t^3_2 - q^{-1} t^2_2 t^3_1) = 1,$$

where $[a, b]_q := ba - qab$ and $\lambda = q - q^{-1}$. The reduced version is denoted by $c_q[SL_3]$ and has the additional relations

$$(t^i_j)^n = \delta_{ij}.$$

Throughout this section we limit ourselves to q a primitive $n = 2m + 1$ -th root of unity so that $\mathfrak{c}_q[SL_2] \cong c_{q^{-m}}[SL_2]$ according to Theorem 4.1. Since we only consider this case, it will be convenient to use the isomorphism to define new generators a, b, c, d of $\mathfrak{c}_q[SL_2]$ related to our previous ones by $X = bd^{-1}, t = d^{-2}$ and $Y = d^{-1}c/(q^m - q^{-m})$. Then we can benefit from both the matrix form of coproduct on the new set and the dual basis feature of the original set. We let $A = \widehat{\mathfrak{c}_q[SL_2]} = \mathfrak{c}_q[SL_2] \otimes \mathbb{C}_q[\varsigma] / (\varsigma^n - 1)$ be the central extension dual to $\widehat{u_q(sl_2)} = u_q(sl_2) \otimes \mathbb{C}_q[g] / (g^n - 1)$. Here $\langle \varsigma, g \rangle = q$ and $\mathcal{R}(\varsigma, \varsigma) = q$ is the coquasitriangular structure on the central extension factor. Let B be a quantum-braided plane \mathfrak{c}_q^2 as in Lemma 6.1 but viewed in the category of left comodules over A with left coaction

$$(6.4) \quad \Delta_L \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \otimes \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}; \quad \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varsigma^{m\beta},$$

where we now denote the generators X_1, X_2 . In this case we will have

$$\Psi(X_i \otimes X_j) = q^{m^2\beta^2} R^j_k{}^i_l X_k \otimes X_l,$$

where R was given in (4.3). We again require that $\beta^2 = 3 \pmod n$ so that $q^{3m^2}R$ has the correct normalisation factor $q^{3m^2+m(m+1)} = q^{-m}$ in front of the matrix in (4.3), as needed to obtain a braided-Hopf algebra. One also has, c.f. [14],

$$\underline{\Delta}(X_1^r X_2^s) = \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{-ms_1(r-r_1)} X_1^{r_1} X_2^{s_1} \otimes X_1^{r-r_1} X_2^{s-s_1}.$$

The dual B^* was likewise explained in the previous section and is now taken with generators Y_i and regarded in the category of right comodules over A with

$$(6.5) \quad \Delta_R \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \otimes \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}.$$

Theorem 6.5. *Let $n = 2m + 1$ such that $\beta^2 = 3$ is solved mod n . Let $A = \widetilde{c_q[SL_2]} = c_q[SL_2] \otimes c_q[\varsigma]/(\varsigma^n - 1)$ regarded with generators $\varsigma, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. Let B, B^* be quantum-braided planes with generators X_i, Y_i for $i = 1, 2$ as above.*

(1) *The co-double bosonisation, denoted $c_q[SL_3]$, has cross relations and coproducts*

$$X_i Y_j = Y_j X_i, \quad X_i \varsigma = q^{m\beta} \varsigma X_i, \quad Y_i \varsigma = q^{m\beta} \varsigma Y_i, \quad \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} X_1 = \begin{pmatrix} q^{-1} X_1 \tilde{a} & q^{-1} X_1 \tilde{b} \\ q^m X_1 \tilde{c} & q^m X_1 \tilde{d} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} X_2 = \begin{pmatrix} q^m X_2 \tilde{a} + (q^{-1} - 1) X_1 \tilde{c} & q^m X_2 \tilde{b} + (q^{-1} - 1) X_1 \tilde{d} \\ q^{-1} X_2 \tilde{c} & q^{-1} X_2 \tilde{d} \end{pmatrix},$$

$$Y_1 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} q \tilde{a} Y_1 & q^{-m} \tilde{b} Y_1 \\ q \tilde{c} Y_1 & q^{-m} \tilde{d} Y_1 \end{pmatrix}, \quad Y_2 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} q^{-m} \tilde{a} Y_2 + (q - 1) \tilde{b} Y_1 & q \tilde{b} Y_2 \\ q^{-m} \tilde{c} Y_2 + (q - 1) \tilde{d} Y_1 & q \tilde{d} Y_2 \end{pmatrix},$$

$$\Delta X_1 = X_1 \otimes 1 + \sum_{r,s=0}^{n-1} (q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q (\tilde{a} Y_1^r Y_2^s \otimes X_1^{r+1} X_2^s + q^{-mr} \tilde{b} Y_1^r Y_2^s \otimes X_1^r X_2^{s+1}),$$

$$\Delta X_2 = X_2 \otimes 1 + \sum_{r,s=0}^{n-1} (q-1)^{r+s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q (\tilde{c} Y_1^r Y_2^s \otimes X_1^{r+1} X_2^s + q^{-mr} \tilde{d} Y_1^r Y_2^s \otimes X_1^r X_2^{s+1}),$$

$$\Delta Y_1 = 1 \otimes Y_1 + \sum_{r,s=0}^{n-1} (q-1)^{r+s-1} q^{-r+ms+1} \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q Y_1^r Y_2^s \otimes X_1^{r-1} X_2^s \tilde{a}$$

$$+ \sum_{r,s=0}^{n-1} (q-1)^{r+s-1} q^{-r-s+1} \begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q Y_1^r Y_2^s \otimes X_1^r X_2^{s-1} \tilde{c},$$

$$\Delta Y_2 = 1 \otimes Y_2 + \sum_{r,s=0}^{n-1} (q-1)^{r+s-1} q^{-r+ms+1} \begin{bmatrix} r+s-1 \\ s \end{bmatrix}_q Y_1^r Y_2^s \otimes X_1^{r-1} X_2^s \tilde{b}$$

$$+ \sum_{r,s=0}^{n-1} (q-1)^{r+s-1} q^{-r-s+1} \begin{bmatrix} r+s-1 \\ s-1 \end{bmatrix}_q Y_1^r Y_2^s \otimes X_1^r X_2^{s-1} \tilde{d},$$

$$\Delta \varsigma = \sum_{r,s=0}^{n-1} q^{-m\beta(r+s)} (q-1)^{r+s} \begin{bmatrix} r+2m\beta-1 \\ r \end{bmatrix}_q \begin{bmatrix} r+s+2m\beta-1 \\ s \end{bmatrix}_q \varsigma Y_1^r Y_2^s \otimes X_1^r X_2^s \varsigma,$$

$$\Delta \tilde{a} = \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{-r-s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{a} Y_1^r Y_2^s \otimes (q^{-ms} [r+1]_q X_1^r X_2^s \tilde{a} + [s]_q X_1^{r+1} X_2^{s-1} \tilde{c})$$

$$+ \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{m(r+s)} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{b} Y_1^r Y_2^s \otimes (q^{-m} [r]_q X_1^{r-1} X_2^{s+1} \tilde{a} + \otimes q^s X_1^r X_2^s \tilde{c}),$$

$$\begin{aligned}
\Delta \tilde{b} &= \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{-r-s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{a} Y_1^r Y_2^s \otimes (q^{-ms} [r+1]_q X_1^r X_2^s \tilde{b} + [s]_q X_1^{r+1} X_2^{s-1} \tilde{d}) \\
&\quad + \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{m(r+s)} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{b} Y_1^r Y_2^s \otimes (q^{-m} [r]_q X_1^{r-1} X_2^{s+1} \tilde{b} + \otimes q^s X_1^r X_2^s \tilde{d}), \\
\Delta \tilde{c} &= \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{-r-s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{c} Y_1^r Y_2^s \otimes (q^{-ms} [r+1]_q X_1^r X_2^s \tilde{a} + [s]_q X_1^{r+1} X_2^{s-1} \tilde{c}) \\
&\quad + \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{m(r+s)} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{d} Y_1^r Y_2^s \otimes (q^{-m} [r]_q X_1^{r-1} X_2^{s+1} \tilde{a} + \otimes q^s X_1^r X_2^s \tilde{c}), \\
\Delta \tilde{d} &= \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{-r-s} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{c} Y_1^r Y_2^s \otimes (q^{-ms} [r+1]_q X_1^r X_2^s \tilde{b} + [s]_q X_1^{r+1} X_2^{s-1} \tilde{d}) \\
&\quad + \sum_{r,s=0}^{n-1} (q-1)^{r+s} q^{m(r+s)} \begin{bmatrix} r+s \\ s \end{bmatrix}_q \tilde{d} Y_1^r Y_2^s \otimes (q^{-m} [r]_q X_1^{r-1} X_2^{s+1} \tilde{b} + \otimes q^s X_1^r X_2^s \tilde{d}).
\end{aligned}$$

(2) If $n > 3$ and is not divisible by 3 then $\mathfrak{c}_q[SL_3]$ is isomorphic to $c_{q^{-m}}[SL_3]$ with its standard coquasitriangular structure.

Proof. (1) From $X_2 \cdot_{\text{op}} X_1 = q X_1 \cdot_{\text{op}} X_2$, we work inductively and find that

$$X_1^{r(\text{op})} \cdot_{\text{op}} X_2^{s(\text{op})} = q^{\frac{-(r+s)(r+s-1)}{2}} X_1^r X_2^s, \quad \bar{S}(X_1^{r(\text{op})} \cdot_{\text{op}} X_2^{s(\text{op})}) = (-1)^{r+s} q^{\frac{-(r+s)(r+s-1)}{2}} X_1 X_2$$

where $X_1^{r(\text{op})}$ means $X_1 \cdot_{\text{op}} X_1 \cdot_{\text{op}} \cdots$ r -times. We also need that

$$\Delta_L(X_1^r) = \sum_{r_1=0}^r \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \tilde{a}^{r_1} \tilde{b}^{r-r_1} \otimes X_1^{r_1} X_2^{r-r_1}, \quad \Delta_L(X_2^s) = \sum_{s_1=0}^s \begin{bmatrix} s \\ s_1 \end{bmatrix}_q \tilde{c}^{r_1} \tilde{d}^{s-s_1} \otimes X_1^{s_1} X_2^{s-s_1}$$

and that ζ commutes with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. Then computation from Theorem 3.1 gives

$$X_i Y_j = Y_j X_i, \quad X_i \zeta = q^{m\beta} \zeta X_i, \quad Y_i \zeta = q^{m\beta} \zeta Y_i, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} X_1 = \begin{pmatrix} q^{-m^2} X_1 a & q^{-m^2} X_1 b \\ q^{m^2} X_1 c & q^{m^2} X_1 d \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X_2 = \begin{pmatrix} q^{m^2} (X_2 a + (q^m - q^{-m}) X_1 c) & q^{m^2} (X_2 b + (q^m - q^{-m}) X_1 d) \\ q^{-m^2} X_2 c & q^{-m^2} X_2 d \end{pmatrix},$$

$$Y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{m^2} a Y_1 & q^{-m^2} b Y_1 \\ q^{m^2} c Y_1 & q^{-m^2} d Y_1 \end{pmatrix}, \quad Y_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-m^2} (a Y_2 - (q^m - q^{-m}) b Y_1) & q^{m^2} b Y_2 \\ q^{-m^2} (c Y_2 - (q^m - q^{-m}) d Y_1) & q^{m^2} d Y_2 \end{pmatrix}$$

and hence the relations stated. The algebra generated by $X_i, Y_i, \zeta, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ is n^8 dimensional as required for these to be all the relations. For the coproduct, we use Lemma 6.2 to provide a basis and dual basis of \mathfrak{c}_q^2 and $(\mathfrak{c}_q^2)^*$. Then

$$\begin{aligned}
\Delta X_1 &= X_1 \otimes 1 \\
&\quad + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1) + \frac{(r_1+s_1)(r_1+s_1+1)}{2} + r_1+s_1} \frac{\tilde{a} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^{r+1} X_2^s \\
&\quad + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1) + \frac{(r_1+s_1)(r_1+s_1-1)}{2} + s_1 - mr} (1 + [r_1]_q (q-1)) \\
&\quad \quad \quad \frac{\tilde{b} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^r X_2^{s+1}
\end{aligned}$$

and similarly for ΔX_2 . Likewise,

$$\Delta Y_1 = 1 \otimes Y_1 + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [r_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1) + \frac{(r_1+s_1-1)(r_1+s_1-2)}{2} - r + ms + r_1 + s_1}$$

$$\begin{aligned} & \frac{Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^{r-1} X_2^s \tilde{a} \\ & + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{r_1+s_1-r-s+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} (-1)^{r+s-r_1-s_1} ([s_1]_q + [r_1]_q [s-s_1]_q) \\ & \frac{Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^r X_2^{s-1} \tilde{c} \end{aligned}$$

and similarly for ΔY_2 . Next, we have

$$\begin{aligned} \Delta \varsigma &= \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q q^{s_1(r-r_1)+m\beta(2r_1+2s_1-r-s)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} (-1)^{r+s-r_1-s_1} \frac{\varsigma Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^r X_2^s \varsigma. \\ \Delta \tilde{a} &= \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+2r_1+s_1-r+ms+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} \\ & \frac{\tilde{a} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^r X_2^s \tilde{a} \\ & + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [s-s_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+2r_1+2s_1-r-s+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} \\ & (1-q) \frac{\tilde{a} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^{r+1} X_2^{s-1} \tilde{c} \\ & + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q [r_1]_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+r_1+s_1+mr+ms+\frac{(r_1+s_1)(r_1+s_1-1)}{2}} \\ & (q^{-m} - q^m) \frac{\tilde{b} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^{r-1} X_2^{s+1} \tilde{a} \\ & + \sum_{r,s=0}^{n-1} \sum_{r_1=0}^r \sum_{s_1=0}^s \begin{bmatrix} r \\ r_1 \end{bmatrix}_q \begin{bmatrix} s \\ s_1 \end{bmatrix}_q (-1)^{r+s-r_1-s_1} q^{s_1(r-r_1)+\frac{(r_1+s_1)(r_1+s_1-1)}{2}+r_1+2s_1+mr-s} \\ & (1 - [r_1]_q [s-s_1]_q (q-1)^2) \frac{\tilde{b} Y_1^r Y_2^s}{[r]_q! [s]_q!} \otimes X_1^r X_2^s \tilde{c} \end{aligned}$$

and similarly for $\Delta \tilde{b}, \Delta \tilde{c}, \Delta \tilde{d}$. The stated coproducts follow from the q -identity

$$(6.6) \quad \sum_{r_1=0}^r \frac{(-1)^{r-r_1} q^{\frac{r_1(r_1+1)}{2}} q^{sr_1}}{[r_1]_q! [r-r_1]_q!} = (q-1)^r \begin{bmatrix} r+s \\ r \end{bmatrix}_q$$

for all r, s (of which (4.2) are special cases) and further calculation.

(2) If $n > 3$ and is not divisible by 3 then β is invertible mod n . We define $\varphi : c_{q^{-m}}[SL_3] \rightarrow c_q[SL_3]$ by

$$\begin{aligned} \varphi(t^1_1) &= a\varsigma^{\frac{m}{\beta}} + \lambda X_1, \quad \varphi(t^1_2) = b\varsigma^{\frac{m}{\beta}} + \lambda X_1, \quad \varphi(t^1_3) = X_1 \varsigma^{\frac{1}{\beta}}, \\ \varphi(t^2_1) &= c\varsigma^{\frac{m}{\beta}} + \lambda X_2, \quad \varphi(t^2_2) = d\varsigma^{\frac{m}{\beta}} + \lambda X_2 \varsigma^{\frac{1}{\beta}} Y_2, \quad \varphi(t^2_3) = X_2 \varsigma^{\frac{1}{\beta}}, \\ \varphi(t^3_1) &= \lambda \varsigma^{\frac{1}{\beta}} Y_1, \quad \varphi(t^3_2) = (q^m - q^{-m}) \varsigma^{\frac{1}{\beta}} Y_2, \quad \varphi(t^3_3) = \varsigma^{\frac{1}{\beta}}, \end{aligned}$$

where $\lambda = q^m - q^{-m}$. A tedious calculation shows that this extends as an algebra map and is a coalgebra map. In the other direction, we define $\phi : c_q[SL_3] \rightarrow c_{q^{-m}}[SL_3]$ by

$$\begin{aligned} \phi(\varsigma) &= (t^3_3)^\beta, \quad \phi(X_1) = t^1_3 (t^3_3)^{-1}, \quad \phi(X_2) = t^2_3 (t^3_3)^{-1}, \\ \phi(Y_1) &= (q^m - q^{-m})^{-1} (t^3_3)^{-1} t^3_1, \quad \phi(Y_2) = (q^m - q^{-m})^{-1} (t^3_3)^{-1} t^3_2, \\ \phi(a) &= t^1_1 (t^3_3)^{-m} - q^m t^1_3 t^3_1 (t^3_3)^m, \quad \phi(b) = t^1_2 (t^3_3)^{-m} - q^m t^1_3 t^3_2 (t^3_3)^m, \\ \phi(c) &= t^2_1 (t^3_3)^{-m} - q^m t^2_3 t^3_1 (t^3_3)^m, \quad \phi(d) = t^2_2 (t^3_3)^{-m} - q^m t^2_3 t^3_2 (t^3_3)^m \end{aligned}$$

as inverse to φ . Although one can verify these matters directly, the map φ was obtained as adjoint to the isomorphism $u_{q^{-m}}(sl_3) \rightarrow u_q(sl_3)$ in part (2) of Lemma 6.3 as follows. The standard duality between $u_{q^{-m}}(sl_3)$ and $c_{q^{-m}}[SL_3]$ is by

$$\begin{aligned} \langle \mathbf{t}, F_1 \rangle &= \mathbf{e}_{12}, & \langle \mathbf{t}, F_2 \rangle &= \mathbf{e}_{23}, & \langle \mathbf{t}, F_{12} \rangle &= \mathbf{e}_{13}, & \langle \mathbf{t}, E_1 \rangle &= \lambda \mathbf{e}_{21}, & \langle \mathbf{t}, E_2 \rangle &= \lambda \mathbf{e}_{32}, \\ \langle \mathbf{t}, E_{12} \rangle &= \lambda \mathbf{e}_{31}, & \langle \mathbf{t}, K_1 \rangle &= \begin{pmatrix} q^{-m} & 0 & 0 \\ 0 & q^m & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \langle \mathbf{t}, K_2 \rangle &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-m} & 0 \\ 0 & 0 & q^m \end{pmatrix}, \end{aligned}$$

where \mathbf{e}_{ij} is an elementary matrix with entry 1 in (i, j) -position and 0 elsewhere. The duality between $u_q(sl_3)$ and $\mathbf{c}_q[SL_3]$ is part of our construction with a natural basis of $\mathbf{c}_q[SL_3]$ built from bases of $\mathbf{c}_q^2, (\mathbf{c}_q^2)^*$ and $\widetilde{\mathbf{c}_q[SL_2]} = \mathbf{c}_q[SL_2] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^n - 1)$. The first tensor factor here has a basis of monomials in X, t, Y by Theorem 4.1. Therefore we have $\{X_1^{i_1} X_2^{i_2} X^{i_3} t^{j_1} \varsigma^{j_2} Y^{k_1} Y_1^{k_2} Y_2^{k_3}\}$ as a basis of $\mathbf{c}_q[SL_3]$ essentially dual to the PBW basis of $u_q(sl_3)$ in the sense

$$\begin{aligned} \langle X_1^{i_1} X_2^{i_2} X^{i_3} t^{j_1} \varsigma^{j_2} Y^{k_1} Y_1^{k_2} Y_2^{k_3}, F_{12}^{i'_1} F_2^{i'_2} F_1^{i'_3} K_1^{j'_1} g^{j'_2} E_1^{k'_1} E_{12}^{k'_2} E_2^{k'_3} \rangle \\ = \delta_{i_1 i'_1} \delta_{i_2 i'_2} \delta_{i_3 i'_3} \delta_{k_1 k'_1} \delta_{k_2 k'_2} \delta_{k_3 k'_3} [i_1]_{q^{-1}}! [i_2]_{q^{-1}}! [i_3]_{q^{-1}}! q^{j_1 j'_1 + j_2 j'_2} [k_1]_q! [k_2]_q! [k_3]_q!. \end{aligned}$$

This is the dual basis result for $u_q(sl_3)$ and $\mathbf{c}_q[SL_3]$ analogous to Corollary 4.3 in the sl_2 case. Hence the coefficients of $\varphi(t^i_j)$ in this basis of $\mathbf{c}_q[SL_3]$ will be picked out by evaluation against the dual basis $F_{12}^{i_1} F_2^{i_2} F_1^{i_3} \delta_{j_1}(K_1) \delta_{j_2}(g) E_1^{k_1} E_{12}^{k_2} E_2^{k_3}$, where $\delta_j(K_1), \delta_j(g)$ are defined as in Corollary 4.3. These values are given by the matrix representation as above except that we still need the matrix representation of g . From Lemma 6.3 we recall that that $u_q(sl_3) \cong u_{q^{-m}}(sl_3)$ with $g \mapsto (K^{-m} K_2)^{\frac{1}{m\beta}}$, hence we have $\langle \mathbf{t}, g \rangle = \text{diag}(q^{\frac{m}{\beta}}, q^{\frac{m}{\beta}}, q^{\frac{1}{\beta}})$. This gives, for example,

$$\begin{aligned} \langle \varphi(t^1_1), F_{12}^{i_1} F_2^{i_2} F_1^{i_3} \delta_{j_1}(K) \delta_{j_2}(g) E_1^{k_1} E_{12}^{k_2} E_2^{k_3} \rangle &= \delta_{i_1,0} \delta_{i_2,0} \delta_{i_3,0} \delta_{j_1,-m} \delta_{j_2,\frac{m}{\beta}} \delta_{k_1,0} \delta_{k_2,0} \delta_{k_3,0} \\ &\quad + \delta_{i_1,0} \delta_{i_2,0} \delta_{i_3,1} \delta_{j_1,m} \delta_{j_2,\frac{m}{\beta}} \delta_{k_1,1} \delta_{k_2,0} \delta_{k_3,0} (q^m - q^{-m}) \\ &\quad + \delta_{i_1,0} \delta_{i_2,1} \delta_{i_3,0} \delta_{j_1,0} \delta_{j_2,\frac{1}{\beta}} \delta_{k_1,0} \delta_{k_2,0} \delta_{k_3,1} (q^m - q^{-m}), \end{aligned}$$

which by summing against the dual basis implies that

$$\varphi(t^1_1) = t^{-m} \varsigma^{\frac{m}{\beta}} + (q^m - q^{-m}) X t^m \varsigma^{\frac{m}{\beta}} Y + (q^m - q^{-m}) X_1 \varsigma^{\frac{1}{\beta}} Y_1.$$

We then convert over to the a, b, c, d generators as discussed.

Finally, the coquasitriangular structure of $\mathbf{c}_q[SL_3]$ computed using Lemma 3.6 and pulled back to the $c_{q^{-m}}[SL_3]$ generators is $\mathcal{R}(\varphi(t^i_j), \varphi(t^k_l)) = R^I_J$, where $I = (i, k)$, $J = (j, l)$ are taken in lexicographic order $(1, 1), (1, 2), \dots, (3, 3)$ and

$$R^I_J = q^{\frac{m}{3}} \begin{pmatrix} q^{-m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q^{-m} - q^m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q^{-m} - q^m & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q^{-m} - q^m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-m} \end{pmatrix},$$

which is the standard coquasitriangular structure on the generators of $c_p[SL_3]$ given in [15] when specialised to the root of unity $p = q^{-m}$. \square

Remark 6.6. In the case (2) of the theorem above, we can identify $\widetilde{\mathbf{c}_q[SL_2]} \cong c_{q^{-m}}[GL_2]$ by sending the four matrix generators of the latter to $\tilde{a} = a\varsigma^{m\beta}, \tilde{b} = b\varsigma^{m\beta}, \tilde{c} = c\varsigma^{m\beta}, \tilde{d} = d\varsigma^{m\beta}$. The q -determinant D maps to $\varsigma^{2m\beta}$. The converse direction is clear since β is invertible mod n when $n > 3$ and not divisible by 3, so we can write $\varsigma = D^{\frac{1}{2m\beta}}$.

Example 6.7. At $q^3 = 1$, \mathfrak{c}_q^2 is already a braided-Hopf algebra in the category of $\mathfrak{c}_q[SL_2]$ -comodules without a central extension. Therefore we can apply Theorem 3.1 and obtain a Hopf algebra, which we denote $\mathfrak{c}_q'[SL_3]$, generated by X_i, Y_i, a, b, c, d with the additional cross relations and coproducts

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X_1 = \begin{pmatrix} q^{-1}X_1a & q^{-1}X_1b \\ qX_1c & qX_1d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} X_2 = \begin{pmatrix} qX_2a + (q^{-1} - 1)X_1c & qX_2b + (q^{-1} - 1)X_1d \\ q^{-1}X_2c & q^{-1}X_2d \end{pmatrix},$$

$$Y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} qaY_1 & q^{-1}bY_1 \\ qcY_1 & q^{-1}dY_1 \end{pmatrix}, \quad Y_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1}aY_2 + (q - 1)bY_1 & qbY_2 \\ q^{-1}cY_2 + (q - 1)dY_1 & qdY_2 \end{pmatrix},$$

$$\begin{aligned} \Delta X_1 &= X_1 \otimes 1 + a \otimes X_1 + b \otimes X_2 + \lambda a Y_1 \otimes X_1^2 + \lambda b Y_2 \otimes X_2^2 + \lambda a Y_2 \otimes X_1 X_2 + \lambda q^2 b Y_1 \otimes X_1 X_2 \\ &\quad + \lambda^2 a Y_2^2 \otimes X_1 X_2^2 + \lambda^2 q b Y_1^2 \otimes X_1^2 X_2 + \lambda^2 [2]_q a Y_1 Y_2 \otimes X_1^2 X_2 + \lambda^2 [2]_q q^2 b Y_1 Y_2 \otimes X_1 X_2^2, \\ \Delta Y_1 &= 1 \otimes Y_1 + Y_1 \otimes a + Y_2 \otimes c + \lambda q^2 Y_1^2 \otimes X_1 a + \lambda q^2 Y_2^2 \otimes X_2 c + \lambda q Y_1 Y_2 \otimes X_2 a + \lambda q^2 Y_1 Y_2 \otimes X_1 c \\ &\quad + \lambda^2 q^2 Y_1 Y_2^2 \otimes X_2^2 a + \lambda^2 [2]_q q Y_1 Y_2^2 \otimes X_1 X_2 c + \lambda^2 [2]_q Y_1^2 Y_2 \otimes X_1 X_2 a + \lambda^2 q Y_1^2 Y_2 \otimes X_1^2 c, \\ \Delta a &= a \otimes a + b \otimes c + \lambda [2]_q q^2 a Y_1 \otimes X_1 a + \lambda q a Y_2 \otimes X_2 a + \lambda q^2 a Y_2 \otimes X_1 c + \lambda [2]_q q^{-1} b Y_2 \otimes X_2 c \\ &\quad + \lambda b Y_1 \otimes X_2 a + \lambda q b Y_1 \otimes X_1 c + \lambda^2 q^2 a Y_2^2 \otimes X_2^2 a + \lambda^2 q^2 b Y_1^2 \otimes X_1^2 c + \lambda^2 [2]_q q a Y_2^2 \otimes X_1 X_2 c \\ &\quad + \lambda^2 [2]_q q b Y_1^2 \otimes X_1 X_2 a + \lambda^2 [2]_q^2 a Y_1 Y_2 \otimes X_1 X_2 a + \lambda^2 [2]_q^2 b Y_1 Y_2 \otimes X_1 X_2 c \\ &\quad + \lambda^2 [2]_q q a Y_1 Y_2 \otimes X_1^2 c + \lambda^2 [2]_q q b Y_1 Y_2 \otimes X_2^2 a, \end{aligned}$$

and similarly for the remaining coproducts. Here $\lambda = q - 1$. This $\mathfrak{c}_q'[SL_3]$ is dual to $\mathfrak{u}'_q(sl_3)$ in Example 6.4 and it is not isomorphic to $c_{q^{-1}}[SL_3]$, but rather to a sub-Hopf algebra by $\phi : \mathfrak{c}_q'[SL_3] \hookrightarrow c_{q^{-1}}[SL_3]$ with

$$\begin{aligned} \phi(X_1) &= t^1_3(t^3_3)^{-1}, \quad \phi(X_2) = t^2_3(t^3_3)^{-1}, \quad \phi(Y_1) = -\frac{t^3_1(t^3_3)^{-1}}{\lambda}, \quad \phi(Y_2) = -\frac{t^3_2(t^3_3)^{-1}}{\lambda}, \\ \phi(a) &= t^1_1(t^3_3)^{-1} - t^1_3(t^3_3)^{-1}t^3_1(t^3_3)^{-1}, \quad \phi(b) = t^1_2(t^3_3)^{-1} - t^1_3(t^3_3)^{-1}t^3_2(t^3_3)^{-1}, \\ \phi(c) &= t^2_1(t^3_3)^{-1} - t^2_3(t^3_3)^{-1}t^3_1(t^3_3)^{-1}, \quad \phi(d) = t^2_2(t^3_3)^{-1} - t^2_3(t^3_3)^{-1}t^3_2(t^3_3)^{-1}. \end{aligned}$$

Moreover, $\mathfrak{c}_q'[SL_3]$ is a coquasitriangular Hopf algebra by Lemma 3.6. Writing $s^1_1 = a, s^1_2 = b, s^2_1 = c, s^2_2 = d$ for the matrix form of the generators of $\mathfrak{c}_q[SL_2]$, the coquasitriangular structure of $\mathfrak{c}_q'[SL_3]$ comes out as

$$\mathcal{R}(s^i_j, s^k_l) = R^i_j{}^k_l, \quad \mathcal{R}(X_i, Y_j) = -\delta_{i,j}, \quad \mathcal{R}(X_i, X_j) = \mathcal{R}(Y_i, Y_j) = \mathcal{R}(Y_i, X_j) = 0,$$

$$\mathcal{R}(X_i, s^j_k) = \mathcal{R}(Y_i, s^j_k) = \mathcal{R}(s^i_j, X_k) = \mathcal{R}(s^i_j, Y_k) = 0,$$

$$\mathcal{R}(X_i s^j_k, s^u_v Y_w) = -\delta_{w_1, i} R^j_{j_1}{}^{w_1}{}_w R^{j_1}{}_k{}^u{}_v, \quad \mathcal{R}(s^i_j Y_k, X_u s^v_w) = 0,$$

where R is as in (4.3) with $m = 1$. Theorem 6.5 (1) still applies at $q^3 = 1$ with $\beta = 0$ giving that ς is central and group-like in $\mathfrak{c}_q[SL_3]$ and that $\mathfrak{c}_q[SL_3] \cong \mathfrak{c}_q'[SL_3] \otimes \mathbb{C}_q[\varsigma]/(\varsigma^3 - 1)$.

6.3. Fermionic version of $\mathbb{C}_q[SL_3]$. Here we similarly apply co-double bosonisation but this time to obtain a part-fermionic version of $\mathbb{C}_q[SL_3]$ by using the fermionic quantum-braided plane. We no longer work at roots of unity but rather with q generic and also, in the spirit of Remark 6.6, we take as our middle Hopf algebra $A = \mathbb{C}_q[GL_2]$, the coquasitriangular Hopf algebra generated by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ with the same q -commutation relations and coalgebra structure as $\mathbb{C}_q[SL_2]$, but with $D = \tilde{a}\tilde{d} - q^{-1}\tilde{b}\tilde{c} = \tilde{d}\tilde{a} - q\tilde{b}\tilde{c}$ inverted. The antipode and coquasitriangular structure are given in matrix form by

$$S \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = D^{-1} \begin{pmatrix} \tilde{d} & -q\tilde{b} \\ -q^{-1}\tilde{c} & \tilde{a} \end{pmatrix}, \quad R = -q^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

In fact the normalisation of R here can be chosen freely (there is a 1-parameter family of such quasitriangular structures on this Hopf algebra) which we have fixed so that we

have $B = \mathbb{C}_q^{0|2} \in {}^A\mathcal{M}$ as a fermionic quantum-braided plane generated by e_1, e_2 with the relations and coproduct and braiding

$$e_i^2 = 0, \quad e_2 e_1 + q^{-1} e_1 e_2 = 0, \quad \underline{\Delta} e_i = e_i \otimes 1 + 1 \otimes e_i, \quad \underline{\epsilon} e_i = 0, \quad \underline{S} e_i = -e_i, \\ \Psi(e_i \otimes e_i) = -e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = -q^{-1} e_2 \otimes e_1, \quad \Psi(e_2 \otimes e_1) = -q^{-1} e_1 \otimes e_2 - (1 - q^{-2}) e_2 \otimes e_1.$$

This has a left $\mathbb{C}_q[GL_2]$ -coaction as in (6.4). Similarly, its dual $B^* = (\mathbb{C}_q^{0|2})^*$ lives in the category of right $\mathbb{C}_q[GL_2]$ -comodules with coaction as in (6.5).

Proposition 6.8. *Let $q \in \mathbb{C}^*$ not be a root of unity. The co-double bosonisation $B^{\text{op}} \bowtie A \bowtie B^*$ with the above B, A, B^* is a coquasitriangular Hopf algebra $\mathbb{C}_q^{fer}[SL_3]$ generated by e_i, f_i for $i = 1, 2$ and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, D, D^{-1}$, with cross relations and coproducts*

$$f_i e_j = e_j f_i, \quad \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} e_1 = \begin{pmatrix} -e_1 \tilde{a} & -e_1 \tilde{b} \\ -q e_1 \tilde{c} & -q e_1 \tilde{d} \end{pmatrix}, \\ \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} e_2 = \begin{pmatrix} -q e_2 \tilde{a} - (1 - q^2) e_1 \tilde{c} & -q e_2 \tilde{b} - (1 - q^2) e_1 \tilde{d} \\ -e_2 \tilde{c} & -e_2 \tilde{d} \end{pmatrix}, \\ f_1 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} -\tilde{a} f_1 & -q^{-1} \tilde{b} f_1 \\ -\tilde{c} f_1 & -q^{-1} \tilde{d} f_1 \end{pmatrix}, \quad f_2 \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} -q^{-1} \tilde{a} f_2 - (1 - q^{-2}) \tilde{b} f_1 & -\tilde{b} f_2 \\ -q^{-1} \tilde{c} f_2 - (1 - q^{-2}) \tilde{d} f_1 & -\tilde{d} f_2 \end{pmatrix}, \\ \Delta e_1 = e_1 \otimes 1 + \tilde{a} \otimes e_1 + \tilde{b} \otimes e_2 + (1 - q^{-2})(q^{-1} \tilde{b} f_1 - \tilde{a} f_2) \otimes e_1 e_2, \\ \Delta e_2 = e_2 \otimes 1 + \tilde{c} \otimes e_1 + \tilde{d} \otimes e_2 + (1 - q^{-2})(q^{-1} \tilde{d} f_1 - \tilde{c} f_2) \otimes e_1 e_2, \\ \Delta f_1 = 1 \otimes f_1 + f_1 \otimes \tilde{a} + f_2 \otimes \tilde{c} + (1 - q^2) f_1 f_2 \otimes (e_1 \tilde{c} - q^{-1} e_2 \tilde{a}), \\ \Delta f_2 = 1 \otimes f_2 + f_1 \otimes \tilde{b} + f_2 \otimes \tilde{d} + (1 - q^2) f_1 f_2 \otimes (e_1 \tilde{d} - q^{-1} e_2 \tilde{b}), \\ \Delta \tilde{a} = \tilde{a} \otimes \tilde{a} + \tilde{b} \otimes \tilde{c} + (q - q^{-1})(\tilde{b} f_1 - q \tilde{a} f_2) \otimes (e_1 \tilde{c} - q^{-1} e_2 \tilde{a}), \\ \Delta \tilde{b} = \tilde{a} \otimes \tilde{b} + \tilde{b} \otimes \tilde{d} + (q - q^{-1})(\tilde{b} f_1 - q \tilde{a} f_2) \otimes (e_1 \tilde{d} - q^{-1} e_2 \tilde{b}), \\ \Delta \tilde{c} = \tilde{c} \otimes \tilde{a} + \tilde{d} \otimes \tilde{c} + (q - q^{-1})(\tilde{d} f_1 - q \tilde{c} f_2) \otimes (e_1 \tilde{c} - q^{-1} e_2 \tilde{a}), \\ \Delta \tilde{d} = \tilde{c} \otimes \tilde{b} + \tilde{d} \otimes \tilde{d} + (q - q^{-1})(\tilde{d} f_1 - q \tilde{c} f_2) \otimes (e_1 \tilde{d} - q^{-1} e_2 \tilde{b}).$$

Proof. First note that

$$\mathcal{R}(S\tilde{a}, \tilde{a}) = \mathcal{R}(S\tilde{d}, \tilde{d}) = -1, \quad \mathcal{R}(S\tilde{a}, \tilde{d}) = \mathcal{R}(S\tilde{d}, \tilde{a}) = -q, \quad \mathcal{R}(S\tilde{b}, \tilde{c}) = -(1 - q^2)$$

and zero on other cases of this form. Then the inverse braiding is

$$\Psi^{-1}(e_1 \otimes e_2) = \mathcal{R}(S e_1^{(\bar{1})}, e_2^{(\bar{1})}) e_2^{(\bar{\infty})} \otimes e_1^{(\bar{\infty})} = -q e_2 \otimes e_1 - (1 - q^2) e_1 \otimes e_2,$$

$$\Psi^{-1}(e_2 \otimes e_1) = \mathcal{R}(S e_2^{(\bar{1})}, e_1^{(\bar{1})}) e_1^{(\bar{\infty})} \otimes e_2^{(\bar{\infty})} = -q e_1 \otimes e_2,$$

with the result that $\bar{S}(e_1 \cdot_{\text{op}} e_2) = q^2 e_1 \cdot_{\text{op}} e_2$ and $e_2 \cdot_{\text{op}} e_1 + q^{-1} e_1 \cdot_{\text{op}} e_2 = 0$ in B^{op} . We now apply the co-double bosonisation theorem. It is easy to see that $f_i e_j \equiv (1 \otimes 1 \otimes f_i)(e_j \otimes 1 \otimes 1) = e_j \otimes 1 \otimes 1 \equiv e_j f_i$. Next, we compute that for any $s^i_j \in \mathbb{C}_q[GL_2]$, where $s^1_1 = \tilde{a}, s^1_2 = \tilde{b}, s^2_1 = \tilde{c}, s^2_2 = \tilde{d}$,

$$s^i_j e_k = e_k^{(\bar{\infty})} (s^i_j)_{(2)} \mathcal{R}(S(s^i_j)_{(1)}, e_k^{(\bar{1})}) = \sum_{l=1}^2 e_k^{(\bar{\infty})} s^l_j \mathcal{R}(S s^i_l, e_k^{(\bar{1})}),$$

$$f_k s^i_j = (s^i_j)_{(1)} f_k^{(\bar{0})} \mathcal{R}(f_k^{(\bar{1})}, (s^i_j)_{(2)}) = \sum_{l=1}^2 s^i_l f_k^{(\bar{0})} \mathcal{R}(f_k^{(\bar{1})}, s^l_j),$$

which comes out as the stated cross relations. Now let

$$\{e_a\} = \{1, e_1, e_2, e_1 e_2\}, \quad \{f^a\} = \{1, f_1, f_2, f_1 f_2\}$$

be a basis and dual basis of B, B^* respectively. Then

$$\Delta e_i = e_i \otimes 1 + \sum_{a=1}^2 e_i^{(\bar{1})}{}_{(1)} f^a \otimes (e_{a(1)}^{(\bar{\infty})} \cdot_{\text{op}} e_i^{(\bar{\infty})} \cdot_{\text{op}} \bar{S} e_{a(2)}) \mathcal{R}(e_{a(1)}^{(\bar{1})}, e_i^{(\bar{1})}{}_{(2)}), \\ \Delta f_i = 1 \otimes f_i + \sum_{a=1}^2 f^a \otimes (e_{a(1)} \cdot_{\text{op}} \bar{S} e_{a(3)}^{(\bar{\infty})}) f_i^{(\bar{1})}{}_{(2)} \mathcal{R}(S f_i^{(\bar{1})}{}_{(1)}, e_{a(3)}^{(\bar{\infty})}) \langle f_i, e_{a(2)} \rangle,$$

$$\Delta s^i_j = \sum_{a,k,l,r=1}^2 s^i_k f^a \otimes (e_{a(1)}^{(\infty)} \cdot_{\text{op}} \bar{S} e_{a(2)}^{(\infty)}) s^r_j \mathcal{R}(e_{a(1)}^{(1)}, s^k_l) \mathcal{R}(S s^l_r, e_{a(2)}^{(1)}),$$

which come out as stated for all $i, j \in \{1, 2\}$. Finally, we let

$$s_1 = (q - q^{-1})(\tilde{b}f_1 - q\tilde{a}f_2), \quad s_2 = (q - q^{-1})(\tilde{d}f_1 - q\tilde{c}f_2),$$

$$t_1 = e_1\tilde{c} - q^{-1}e_2\tilde{a}, \quad t_2 = e_1\tilde{d} - q^{-1}e_2\tilde{b},$$

and write $\mathbb{C}_q^{fer}[SL_3]$ as having a matrix of generators t^i_j , where now $i, j \in \{1, 2, 3\}$, by

$$\mathbf{t} = \begin{pmatrix} t^1_1 & t^1_2 & t^1_3 \\ t^2_1 & t^2_2 & t^2_3 \\ t^3_1 & t^3_2 & t^3_3 \end{pmatrix} = \begin{pmatrix} X & t_1 & t_2 \\ s_1 & \tilde{a} & \tilde{b} \\ s_2 & \tilde{c} & \tilde{d} \end{pmatrix};$$

$$(6.7) \quad X = D + (t_1 D^{-1}(\tilde{d}s_1 - q\tilde{b}s_2) - q^{-1}t_2 D^{-1}(\tilde{c}s_1 - q\tilde{a}s_2)).$$

Here D obeys $Dt_i = qt_i D$ and $Ds_i = qs_i D$ for $i = 1, 2$. The coproduct now has the standard matrix form $\Delta \mathbf{t} = \mathbf{t} \otimes \mathbf{t}$ and in these terms the quadratic relations are

$$(t^1_2)^2 = (t^1_3)^2 = (t^2_1)^2 = (t^3_1)^2 = 0,$$

$$[t^1_2, t^1_1]_{q^{-1}} = [t^1_3, t^1_1]_{q^{-1}} = [t^2_1, t^1_1]_{q^{-1}} = [t^3_1, t^1_1]_{q^{-1}} = [t^2_3, t^2_2]_q = [t^3_2, t^2_2]_q = 0,$$

$$[t^3_3, t^2_3]_q = [t^3_3, t^3_2]_q = [t^2_1, t^1_2] = [t^2_1, t^1_3] = [t^3_1, t^1_2] = [t^3_1, t^1_3] = [t^3_2, t^2_3] = 0,$$

$$[t^2_2, t^1_1] = -\lambda t^1_2 t^2_1, \quad [t^2_3, t^1_1] = -\lambda t^1_3 t^2_1, \quad [t^3_2, t^1_1] = -\lambda t^1_2 t^3_1,$$

$$[t^3_3, t^1_1] = -\lambda t^1_3 t^3_1, \quad [t^3_3, t^2_2] = \lambda t^2_3 t^3_2, \quad \{t^1_3, t^1_2\}_{q^{-1}} = \{t^3_1, t^2_1\}_{q^{-1}} = 0,$$

$$\{t^2_2, t^1_2\}_q = \{t^2_2, t^2_1\}_q = \{t^2_3, t^1_3\}_q = \{t^2_3, t^2_1\}_q = \{t^3_2, t^1_2\}_q = \{t^3_2, t^3_1\}_q = 0,$$

$$\{t^3_3, t^1_3\}_q = \{t^3_3, t^3_1\}_q = \{t^2_2, t^1_3\} = \{t^3_1, t^2_2\} = \{t^3_1, t^2_3\} = \{t^3_2, t^1_3\} = 0,$$

$$\{t^2_3, t^1_2\} = -\lambda t^1_3 t^2_2, \quad \{t^3_3, t^1_2\} = -\lambda t^1_3 t^3_2, \quad \{t^3_2, t^2_1\} = \lambda t^2_2 t^3_1, \quad \{t^3_3, t^2_1\} = \lambda t^2_3 t^3_1,$$

where $[\cdot, \cdot]_q$ is as before, similarly $\{a, b\}_q = ab + qba$ for any a, b is the q -anti-commutator, and $\lambda = q - q^{-1}$. Using Lemma 3.6, the values $\mathcal{R}(t^i_j, t^k_l)$ of the coquasitriangular structure of $\mathbb{C}_q^{fer}[SL_3]$ come out, in the same conventions as in the proof of part (2) of Theorem 6.5, as

$$R^I_J = \begin{pmatrix} q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & -q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 \\ 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 & -q^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that since t^1_1 was defined in terms of the other generators including the q -subdeterminant $D = t^2_2 t^3_3 - q^{-1} t^2_3 t^3_2$, there are in fact only 8 algebra generators and 28 q -(anti)commutation relations other than the nilpotency ones and those involving t^1_1 , putting this conceptually on a par with $\mathbb{C}_q[SL_3]$. Instead of a cubic q -determinant relation, we can regard (6.7) as the cubic-quartic relation

$$Dt^1_1 - qt^1_2(t^3_3 t^2_1 - qt^2_3 t^3_1) + t^1_3(t^3_2 t^2_1 - qt^2_2 t^3_1) = D^2.$$

Also note that (2.2) in the ‘R-matrix’ form $R^i_m{}^k_n t^m_j t^n_l = t^k_n t^i_m R^m_j{}^n_l$ (sum over repeated indices) encodes exactly the quadratic relations above for $\mathbb{C}_q^{fer}[SL_3]$ including the nilpotent ones. \square

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